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# INVESTIGATING THE INFINITE SPIDER'S WEB IN COMPLEX DYNAMICS

submitted by

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for the degree of

DOCTOR OF PHILOSOPHY

to

THE OPEN UNIVERSITY

DEPARTMENT OF MATHEMATICS AND STATISTICS

27 September 2012

JOHN OSBORNE

INVESTIGATING THE INFINITE SPIDER'S WEB IN  
COMPLEX DYNAMICS

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*Investigating the infinite spider's web in complex dynamics*

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Youth can not know how age thinks and feels.  
But old men are guilty if they forget what it was to be young.

— J.K. Rowling, *Harry Potter and the Order of the Phoenix*

Dedicated to my family,  
for allowing me to pursue an almost forgotten dream.

## DECLARATION

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This thesis is the result of my own work, except where explicit reference is made to the work of others, and has not been submitted for another qualification to this or any other university.

John Osborne

27 September 2012

## ABSTRACT

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This thesis contains a number of new results on the topological and geometric properties of certain invariant sets in the dynamics of entire functions, inspired by recent work of Rippon and Stallard.

First, we explore the intricate structure of the spider's web fast escaping sets associated with certain transcendental entire functions. Our results are expressed in terms of the components of the complement of the set (the 'holes' in the web). We describe the topology of such components and give a characterisation of their possible orbits under iteration. We show that there are uncountably many components having each of a number of orbit types, and we prove that components with bounded orbits are quasiconformally homeomorphic to components of the filled Julia set of a polynomial. We prove that there are singleton periodic components and that these are dense in the Julia set.

Next, we investigate the connectedness properties of the set of points  $K(f)$  where the iterates of an entire function  $f$  are bounded. We describe a class of transcendental entire functions for which  $K(f)$  is totally disconnected if and only if each component of  $K(f)$  containing a critical point is aperiodic. Moreover we show that, for such functions, if  $K(f)$  is disconnected then it has uncountably many components. We give examples of functions for which  $K(f)$  is totally disconnected, and we use quasiconformal surgery to construct a function for which  $K(f)$  has a component with empty interior that is not a singleton.

Finally we show that, if the Julia set of a transcendental entire function is locally connected, then it must take the form of a spider's web. In the opposite direction, we prove that a spider's web Julia set is always locally connected at a dense subset of buried points. We also show that the set of buried points (the residual Julia set) can be a spider's web.

## PUBLICATIONS

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Much of the content of this thesis has previously been published in the form of papers, as follows:

- (1) The results on the structure of spider's web fast escaping sets (Theorems 1.2 to 1.7 and the contents of Chapter 3) have appeared in the *Bulletin of the London Mathematical Society* [62].
- (2) The material on connectedness properties of the set of points where the iterates of an entire function are bounded (Theorems 1.8 to 1.14 and the contents of Chapter 4) is to appear in the *Mathematical Proceedings of the Cambridge Philosophical Society* [64].
- (3) The results on spiders' webs and locally connected Julia sets of transcendental entire functions (Theorems 1.15 to 1.18 and the contents of Chapter 5) have been published in *Ergodic Theory and Dynamical Systems* [63].



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---

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I thank the Open University for giving me the opportunity to study for a research degree in mathematics at a time in life when many no longer think of themselves as students.

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Finally, and most of all, I thank my family - my wife Sally, my children Pete, Tim, Katie and Rachel, my daughter-in-law Lindsay, my grandchildren Raewyn and Theo and my new son-in-law Jonathan - each of whom has, in various ways, made sacrifices so that I could spend many hours in the pursuit of this project. This thesis is dedicated to them.

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## INTRODUCTION AND MAIN RESULTS

---

The material in this thesis falls within the area of mathematics now known as one-dimensional complex dynamics, with a particular emphasis on the iteration of transcendental entire functions.

The origins of complex dynamics lie in the studies of Schröder, Koenigs, Leau, Böttcher and others on solutions of certain functional equations, but the subject came of age with the research of Fatou and Julia in the early decades of the twentieth century. Making use of Montel's seminal work on normal families of analytic functions (see [57], or [80] for a modern treatment), Fatou and Julia developed an elegant theory which forms the foundation of the subject to this day. Their investigations dealt mainly with the iteration of rational functions, though Fatou also studied transcendental entire functions [37]. An account of the early history of complex dynamics, and references to the original papers, may be found in [3].

*A brief history of  
complex dynamics*

In recognition of the pioneers in the field, the main objects of study in complex dynamics are now known as the Fatou set and the Julia set. Denoting the  $n$ th iterate of a non-linear entire function  $f$  by  $f^n$ ,  $n = 0, 1, 2, \dots$ , the *Fatou set*  $F(f)$  is defined to be the set of points  $z \in \mathbb{C}$  such that the family of functions  $\{f^n : n \in \mathbb{N}\}$  is normal in some neighbourhood of  $z$ . The *Julia set*  $J(f)$  is the complement of  $F(f)$ . Loosely speaking, the dynamical behaviour of  $f$  is stable on the Fatou set and chaotic on the Julia set.

We assume that the reader is familiar with the properties of these sets and the main ideas of one dimensional complex dynamics. For convenience, however, we briefly define the main terms as they arise in this introduction, and also summarise our notation and give some relevant background in Section 2.1. For further details we refer to [14, 24, 56] for rational functions and to [15, 59] for transcendental entire functions.

After its initial flowering, the field of complex dynamics remained largely dormant for five decades. There were exceptions, however, and amongst these we highlight the research of Noel Baker, who published extensively on the subject from the mid-1950s onwards, including many new results on the iteration of transcendental entire functions (see [70] for an appreciation of his life and work). We shall have many occasions to refer to Baker's results in this thesis.

Then, during the 1980s, complex dynamics again became a vibrant area of mathematical research, and this continues to be the case today. This resurgence of interest can be traced to two main factors. The first was the introduction of new techniques from other areas of mathematics that led to the solution of long-standing problems, as in Sullivan's use of quasiconformal mappings to prove that rational functions have no wandering domains, that is, no components of the Fatou set that are not eventually periodic [85]. The second was the advent of computer technology of sufficient power to enable some of the beauty of complex dynamics to be visualised for the first time. This is exemplified by the research of Douady and Hubbard on the iteration of quadratic polynomials [31], given visual expression by Mandelbrot's pictures of the set that now bears his name.

*The escaping set and  
Rippon and  
Stallard's work on  
the fast escaping set*

In addition to the Fatou and Julia sets, another set of considerable importance in understanding the dynamics of a function  $f$  is the *escaping set*,

$$I(f) = \{z \in \mathbb{C} : f^n(z) \rightarrow \infty \text{ as } n \rightarrow \infty\}.$$

The escaping set of a general transcendental entire function was first studied by Eremenko [33], who proved that  $I(f) \cap J(f) \neq \emptyset$ , that  $J(f) = \partial I(f)$  and that every component of  $\overline{I(f)}$  is unbounded. Eremenko also conjectured that every component of  $I(f)$  is unbounded, and this conjecture (which remains open) has stimulated much subsequent research. Because of its importance in this thesis, we give further background on the escaping set of a transcendental entire function in Section 2.2.

The work presented in this thesis takes its inspiration from recent research on the escaping set by Rippon and Stallard [71, 73]. For a



transcendental entire function  $f$ , they studied the *fast escaping set*  $A(f)$ , a subset of  $I(f)$  introduced by Bergweiler and Hinkkanen in [19] and defined in [73] as follows:

$$A(f) = \{z \in \mathbb{C} : \exists \ell \in \mathbb{N} \text{ such that } |f^{n+\ell}(z)| \geq M^n(R, f), \text{ for } n \in \mathbb{N}\}.$$

Here,

$$M(r, f) = \max_{|z|=r} |f(z)|, \text{ for } r > 0,$$

$M^n(r, f)$  is obtained by iterating the function  $r \mapsto M(r, f)$   $n$  times and  $R > 0$  is chosen so that  $M(r, f) > r$  for  $r \geq R$  (for brevity in what follows, we do not repeat this restriction on  $R$  except in formal statements of results, but it should always be assumed to apply). Roughly speaking, the fast escaping set contains those points whose iterates eventually escape to infinity about as fast as is possible, in the sense that they keep ahead of the iterates of the maximum modulus function. The set  $A(f)$  has stronger properties than  $I(f)$ , and it now plays an important role in the study of the dynamics of transcendental functions.

In [71, Theorem 1] Rippon and Stallard proved that, for a general transcendental entire function, every component of the fast escaping set  $A(f)$  is unbounded, which is a partial result in the direction of Eremenko's conjecture. They also showed [71, Theorem 2] that, for a transcendental entire function  $f$  for which  $F(f)$  has a multiply connected component, both  $A(f)$  and  $I(f)$  are connected and contain the closures of all such Fatou components. In view of Baker's result on the properties of multiply connected Fatou components for transcendental entire functions (Lemma 2.7 in Section 2.2), this showed that the escaping set takes a novel form, in striking contrast to the Cantor bouquet structure observed in the escaping sets of many functions in the Eremenko-Lyubich class  $\mathcal{B}$  (see Section 2.2 for further discussion on this point).

These ideas were considerably amplified in [73], a comprehensive study of the set  $A(f)$  for a transcendental entire function  $f$  which included many new results on its properties (we summarise the main results used in this thesis in Sections 2.2 and 3.1). In this paper, Rip-

pon and Stallard first used the term *spider's web* for the new form of the escaping set.

*Definition of a spider's web*

DEFINITION 1.1. A set  $E \subset \mathbb{C}$  is an (infinite) *spider's web* if  $E$  is connected and there exists a sequence  $(G_n)$  of bounded, simply connected domains such that

- $G_{n+1} \supset G_n$ , for  $n \in \mathbb{N}$ ,
- $\partial G_n \subset E$ , for  $n \in \mathbb{N}$  and
- $\bigcup_{n \in \mathbb{N}} G_n = \mathbb{C}$ .

The elements of the sequence  $(\partial G_n)_{n \in \mathbb{N}}$  are sometimes referred to as *loops* in the spider's web.

In [73], Rippon and Stallard also defined the set

$$A_R(f) = \{z \in \mathbb{C} : |f^n(z)| \geq M^n(R, f), \text{ for } n \in \mathbb{N}\},$$

and showed that if  $A_R(f)^c$  has a bounded component, then each of  $A_R(f)$ ,  $A(f)$  and  $I(f)$  is a spider's web [73, Theorem 1.4]. It transpires that  $A_R(f)$  is a spider's web for a wide range of transcendental entire functions [73, Theorem 1.9], [55], [83]. Moreover, if  $A_R(f)$  is a spider's web and  $f$  has no multiply connected Fatou components, then  $J(f)$  is also a spider's web [73, Theorem 1.5] (if  $f$  has a multiply connected Fatou component then  $A_R(f)$  is a spider's web by [73, Theorem 1.9(a)] but  $J(f)$  is disconnected).

*The main idea pursued in this thesis*

The idea that, for certain transcendental entire functions, the sets  $A_R(f)$ ,  $A(f)$ ,  $I(f)$  and  $J(f)$  can take the form of a spider's web is central to the research presented in this thesis. We pursue this idea in three directions:

- We explore the topological and geometric properties of the set  $A(f)$  when  $A_R(f)$  is a spider's web. Our results are expressed in terms of the components of  $A(f)^c$ , the 'holes' in the  $A(f)$  spider's web, and show that  $A(f)$  then has an extremely intricate dynamical structure.
- We present some results on the connectedness properties of the set of points  $K(f)$  where the iterates of an entire function are

bounded. Some of these results are generalisations to a wider class of functions of results already proved for components of  $A(f)^c$  when  $A_R(f)$  is a spider's web.

- We investigate an unexpected link between the spider's web form of the Julia set of a transcendental entire function, and the property of local connectedness.

In the paragraphs that follow, we state our main results and set them in the context of earlier work. Proofs and examples are given in subsequent chapters.

When  $A_R(f)$  is a spider's web, many strong dynamical properties hold. For example, in [73, Theorem 1.6], it is shown that, in this case,

- every component of  $A(f)^c$  is compact, and
- every point of  $J(f)$  is the limit of a sequence of points, each of which lies in a distinct component of  $A(f)^c$ .

We give several further results on the components of  $A(f)^c$  when  $A_R(f)$  is a spider's web. We explore the topological and dynamical properties of components of  $A(f)^c$  and show that, by adapting known results about the components of  $J(f)$  when  $f$  has a multiply connected Fatou component, we are able to obtain new results about the components of  $A(f)^c$  for the wider class of functions where  $A_R(f)$  is a spider's web.

Our first theorem is of a topological nature. Here and elsewhere in the thesis we say that a set  $S \subset \mathbb{C}$  *surrounds* a set or a point if that set or point lies in a bounded complementary component of  $S$ . Recall that a *buried point* of  $J(f)$  is a point of  $J(f)$  that does not lie on the boundary of any Fatou component, and a *buried component* of  $J(f)$  is a component of  $J(f)$  consisting entirely of such buried points. The set of all buried points in  $J(f)$  is called the *residual Julia set* and is denoted by  $J_r(f)$  (see [9, 29] for the properties of this set).

**THEOREM 1.2.** *Let  $f$  be a transcendental entire function, let  $R > 0$  be such that  $M(r, f) > r$  for  $r \geq R$  and let  $A_R(f)$  be a spider's web. Let  $K$  be a component of  $A(f)^c$ . Then:*

*Results on  
components of  
 $A(f)^c$  when  $A_R(f)$   
is a spider's web*

*Topological  
properties of  
components  
of  $A(f)^c$*

- (a)  $\partial K \subset J(f)$  and  $\text{int}(K) \subset F(f)$ . In particular,  $\overline{U} \subset K$  for every Fatou component  $U$  for which  $K \cap \overline{U} \neq \emptyset$ .
- (b) Every neighbourhood of  $K$  contains a closed subset of  $A(f) \cap J(f)$  surrounding  $K$ . If  $K$  has empty interior, then  $K$  consists of buried points of  $J(f)$ .
- (c) If  $f$  has a multiply connected Fatou component, then every neighbourhood of  $K$  contains a multiply connected Fatou component surrounding  $K$ . If, in addition,  $K$  has empty interior, then  $K$  is a buried component of  $J(f)$ .

Note that, if  $A_R(f)$  is a spider's web, then  $f$  maps any component  $K$  of  $A(f)^c$  onto another such component (see Theorem 3.6 in Section 3.2). We call the sequence of iterates of  $K$  its *orbit*, and any infinite subsequence of its iterates a *suborbit*. If  $f^p(K) = K$  for some  $p \in \mathbb{N}$ , then we say that  $K$  is a *periodic component* of  $A(f)^c$ . If  $f^m(K) \neq f^n(K)$  for all  $m > n \geq 0$ , then we say that  $K$  is a *wandering component* of  $A(f)^c$ .

Next, we give a characterisation of the orbits of the components of  $A(f)^c$  when  $A_R(f)$  is a spider's web. To do this, we show how we can use a natural partition of the plane to associate with each component of  $A(f)^c$  a unique 'itinerary' that captures information about its orbit. This enables us to prove the following.

$A(f)^c$  has  
uncountably many  
components

**THEOREM 1.3.** *Let  $f$  be a transcendental entire function, let  $R > 0$  be such that  $M(r, f) > r$  for  $r \geq R$ , and let  $A_R(f)$  be a spider's web. Then  $A(f)^c$  has uncountably many components*

- (a) *whose orbits are bounded,*
- (b) *whose orbits are unbounded but contain a bounded suborbit, and*
- (c) *whose orbits escape to infinity.*

Since there are only countably many Fatou components, we have the following corollary of Theorem 1.2(b) and Theorem 1.3.

**COROLLARY 1.4.** *Let  $f$  be a transcendental entire function, let  $R > 0$  be such that  $M(r, f) > r$  for  $r \geq R$ , and let  $A_R(f)$  be a spider's web. Then the residual Julia set of  $f$  is not empty.*

For our next result, we restrict our attention to those components of  $A(f)^c$  whose orbits are bounded. The proof of Theorem 1.5 uses a technique similar to that adopted by Kisaka in [46]; we describe Kisaka's and other related results in Section 3.5.

**THEOREM 1.5.** *Let  $f$  be a transcendental entire function, let  $R > 0$  be such that  $M(r, f) > r$  for  $r \geq R$ , and let  $A_R(f)$  be a spider's web. Let  $K$  be a component of  $A(f)^c$  whose orbit is bounded. Then there exists a non-linear polynomial  $g$  such that each component of  $A(f)^c$  in the orbit of  $K$  is quasi-conformally homeomorphic to a component of the filled Julia set of  $g$ .*

*Components of  $A(f)^c$  with bounded orbits*

The existence of the quasiconformal mapping in Theorem 1.5 enables us to use recent results from polynomial dynamics [65, 76, 77] to say more about the nature of the components of  $A(f)^c$  whose orbits are bounded.

**THEOREM 1.6.** *Let  $f$  be a transcendental entire function, let  $R > 0$  be such that  $M(r, f) > r$  for  $r \geq R$ , and let  $A_R(f)$  be a spider's web.*

(a) *Let  $K$  be a component of  $A(f)^c$  with bounded orbit. Then:*

(i) *The component  $K$  is a singleton if and only if the orbit of  $K$  includes no periodic component of  $A(f)^c$  containing a critical point. In particular, if  $K$  is a wandering component of  $A(f)^c$ , then  $K$  is a singleton.*

(ii) *The interior of  $K$  is either empty or consists of non-wandering Fatou components. If these Fatou components are not Siegel discs, then they are Jordan domains.*

(b) *All except at most countably many of the components of  $A(f)^c$  with bounded orbits are singletons.*

Note that, by Theorem 1.6(a)(i), if all of the critical points of  $f$  have unbounded orbits (for example, if they all lie in  $I(f)$ ), then every component of  $A(f)^c$  with bounded orbit is a singleton.

Evidently, periodic components of  $A(f)^c$  have bounded orbits, so Theorems 1.5 and 1.6 apply to them in particular. Our final theorem on components of  $A(f)^c$  is a further result on periodic components.

Domínguez [28] has shown that, if  $f$  is a transcendental entire function with a multiply connected Fatou component, then  $J(f)$  has buried



singleton components, and such components are dense in  $J(f)$  (see also [29]). Bergweiler [16] has given an alternative proof of this result, using a method involving the construction of a singleton component of  $J(f)$  which is also a repelling periodic point of  $f$ .

By using a method similar to Bergweiler's, together with our earlier results, we are able to prove the following.

*Singleton periodic  
components  
of  $A(f)^c$*

**THEOREM 1.7.** *Let  $f$  be a transcendental entire function, let  $R > 0$  be such that  $M(r, f) > r$  for  $r \geq R$ , and let  $A_R(f)$  be a spider's web. Then  $A(f)^c$  has singleton periodic components, and such components are dense in  $J(f)$ . If  $f$  has a multiply connected Fatou component, then these singleton periodic components of  $A(f)^c$  are buried components of  $J(f)$ .*

Note that, if  $f$  is a transcendental entire function with a multiply connected Fatou component, then we have shown that singleton periodic components of  $J(f)$  are dense in  $J(f)$ , a slight strengthening of the results of Domínguez [28] and Bergweiler [16].

The first part of Theorem 1.7 is also a strengthening of Rippon and Stallard's result [73, Theorem 1.6] that, if  $A_R(f)$  is a spider's web, then every point in  $J(f)$  is the limit of a sequence of points, each of which lies in a distinct component of  $A(f)^c$ . Note that, by Theorem 1.2(b), if  $A_R(f)$  is a spider's web, then any singleton component of  $A(f)^c$  must be a buried point of  $J(f)$ , but if  $f$  does not have a multiply connected Fatou component, then such a component of  $A(f)^c$  is not a buried component of  $J(f)$ , because  $J(f)$  is a spider's web by [73, Theorem 1.5] and so is connected.

*Results on the  
connectedness  
properties of the set  
where the iterates of  
an entire function  
are bounded*

Our next group of results concerns the set  $K(f)$  of points whose orbits are bounded under iteration,

$$K(f) = \{z \in \mathbb{C} : (f^n(z))_{n \in \mathbb{N}} \text{ is bounded}\}.$$

This set has been much studied where  $f$  is a non-linear polynomial but has received less attention where  $f$  is transcendental entire. Note that, if  $A_R(f)$  is a spider's web, then  $K$  is a component of  $K(f)$  if and only if  $K$  is a component of  $A(f)^c$  with bounded orbit, so Theorems 1.5 and 1.6 are results about  $K(f)$  (see the remark in the introduction to Chapter 4). However, in what follows, we consider a wider class

of transcendental entire functions than those for which  $A_R(f)$  is a spider's web.

If  $f$  is a non-linear polynomial, then  $K(f)$  is a compact set called the *filled Julia set* of  $f$ , and we have  $J(f) = \partial K(f)$  and  $K(f) = \mathbb{C} \setminus I(f)$ . If  $f$  is a transcendental entire function, then it remains true that  $J(f) = \partial K(f)$  (since  $K(f)$  is completely invariant and any Fatou component that meets  $K(f)$  lies in  $K(f)$ ), but  $K(f)$  is not closed or bounded and is not the complement of  $I(f)$ . Indeed, there are always points in  $J(f)$  that are in neither  $I(f)$  nor  $K(f)$  [9, Lemma 1], and there may also be points in  $F(f)$  with the same property [34, Example 1].

Bergweiler [18, Theorem 2] has recently shown that there exist transcendental entire functions for which the Hausdorff dimension of  $K(f)$  is arbitrarily close to 0, whilst Bishop [22] has constructed a transcendental entire function for which, in addition, the Hausdorff dimension of  $J(f)$  is equal to 1. These results are perhaps surprising given that Barański, Karpińska and Zdunik [12] have shown that the Hausdorff dimension of  $K(f) \cap J(f)$  is strictly greater than 1 when  $f$  is in the Eremenko-Lyubich class  $\mathcal{B}$ .

In this light, it is natural to ask questions about the topological nature of  $K(f)$  where  $f$  is transcendental entire, and here we explore some of its connectedness properties. In particular, we give some results on the number of components of  $K(f)$ , and we exhibit a class of transcendental entire functions for which  $K(f)$  is totally disconnected if and only if each component of  $K(f)$  containing a critical point is aperiodic, that is, not periodic.

It is well known that, if  $f$  is a non-linear polynomial and  $K(f)$  contains all of the finite critical points of  $f$ , then both  $J(f)$  and  $K(f)$  are connected, whilst if at least one finite critical point belongs to  $\mathbb{C} \setminus K(f)$  then each of  $J(f)$  and  $K(f)$  has uncountably many components; see, for example, Milnor [56, Theorem 9.5].

For a general transcendental entire function, Baker and Domínguez have shown that  $J(f)$  is either connected or has uncountably many components [8, Theorem B], but no corresponding result is known for  $K(f)$ . However, a result of Rippon and Stallard [74, Theorem 5.2] easily gives the following.

*$K(f)$  is either connected or has infinitely many components*

**THEOREM 1.8.** *Let  $f$  be a transcendental entire function. Then  $K(f)$  is either connected or has infinitely many components.*

A simple example of a function for which  $K(f)$  is connected is the exponential function

$$f(z) = \lambda e^z, \text{ where } 0 < \lambda < 1/e.$$

Recall that, for this function,  $F(f)$  consists of the immediate basin of an attracting fixed point, so that  $F(f) \subset K(f)$ . Since  $F(f)$  is connected and  $\overline{F(f)} = \mathbb{C}$ , it follows that  $K(f)$  is also connected.

At the other extreme, in Section 4.4 we give several examples of functions for which  $K(f)$  is totally disconnected, including the function

$$f(z) = z + 1 + e^{-z},$$

first studied by Fatou (see Example 4.11).

Our next theorem gives a new result on components of  $K(f) \cap J(f)$  for a general transcendental entire function, and a stronger result than Theorem 1.8 on the components of  $K(f)$  for a particular class of functions which we now define.

*Definition of  
strongly  
polynomial-like  
functions*

**DEFINITION 1.9.** We say that a transcendental entire function  $f$  is *strongly polynomial-like* if there exist sequences  $(V_n), (W_n)$  of bounded, simply connected domains with smooth boundaries such that

$$V_n \subset V_{n+1} \text{ and } W_n \subset W_{n+1} \text{ for } n \in \mathbb{N},$$

$$\bigcup_{n \in \mathbb{N}} V_n = \bigcup_{n \in \mathbb{N}} W_n = \mathbb{C},$$

and each of the triples  $(f; V_n, W_n)$  is a polynomial-like mapping in the sense of Douady and Hubbard [32].

We prove the following.

*Connectedness  
properties of*

$K(f) \cap J(f)$  and  $K(f)$

**THEOREM 1.10.** *Let  $f$  be a transcendental entire function.*

- (a) *Either  $K(f) \cap J(f)$  is connected, or else every neighbourhood of a point in  $J(f)$  meets uncountably many components of  $K(f) \cap J(f)$ .*

- (b) If  $f$  is strongly polynomial-like then either  $K(f)$  is connected, or else every neighbourhood of a point in  $J(f)$  meets uncountably many components of  $K(f)$ .

It is also well known from polynomial dynamics that, if  $f$  is a non-linear polynomial, then  $K(f)$  is totally disconnected if all of the critical points of  $f$  lie outside  $K(f)$ ; see for example [24, p. 67]. More generally, Kozlovski and van Strien [48] and Qiu and Yin [65] have recently (and independently) proved results that imply the Branner-Hubbard conjecture, which says that, for a non-linear polynomial  $f$ ,  $K(f)$  is totally disconnected if and only if each component of  $K(f)$  containing a critical point is aperiodic. Indeed, in this case a component of  $K(f)$  is a singleton if and only if its orbit includes no periodic component containing a critical point.

It is natural to ask whether some similar result might hold for certain transcendental entire functions. Using the Branner-Hubbard conjecture, we prove the following theorem which shows that this is the case if  $f$  is strongly polynomial-like. Note that this result is a generalisation of Theorem 1.6 to strongly polynomial-like functions.

**THEOREM 1.11.** *Let  $f$  be a strongly polynomial-like transcendental entire function and let  $K$  be a component of  $K(f)$ .*

*Components of  $K(f)$   
when  $f$  is strongly  
polynomial-like*

- (a) *The component  $K$  is a singleton if and only if the orbit of  $K$  includes no periodic component of  $K(f)$  containing a critical point. In particular, if  $K$  is a wandering component of  $K(f)$ , then  $K$  is a singleton.*
- (b) *The interior of  $K$  is either empty or consists of bounded, non-wandering Fatou components. If these Fatou components are not Siegel discs, then they are Jordan domains.*

**COROLLARY 1.12.** *Let  $f$  be a strongly polynomial-like transcendental entire function.*

- (a) *All except at most countably many components of  $K(f)$  are singletons.*
- (b)  *$K(f)$  is totally disconnected if and only if each component of  $K(f)$  containing a critical point is aperiodic.*

*Alternative  
characterisation  
of strongly  
polynomial-like  
functions*

The following alternative characterization of strongly polynomial-like functions is useful for checking that functions have this property and may be of independent interest.

**THEOREM 1.13.** *A transcendental entire function  $f$  is strongly polynomial-like if and only if there exists a sequence of bounded, simply connected domains  $(D_n)_{n \in \mathbb{N}}$  such that*

- $\overline{D}_n \subset D_{n+1}$ , for  $n \in \mathbb{N}$ ,
- $\bigcup_{n \in \mathbb{N}} D_n = \mathbb{C}$ , and
- $f(\partial D_n)$  surrounds  $\overline{D}_n$ , for  $n \in \mathbb{N}$ .

The next theorem shows that there are large classes of transcendental entire functions which have the property of being strongly polynomial-like. The terminology used in this theorem is explained in Section 4.3.

*Types of function  
that are strongly  
polynomial-like*

**THEOREM 1.14.** *A transcendental entire function  $f$  is strongly polynomial-like if there exists an unbounded sequence  $(r_n)$  of positive real numbers such that*

$$m(r_n, f) := \min\{|f(z)| : |z| = r_n\} > r_n, \quad \text{for } n \in \mathbb{N}.$$

*In particular, this is the case if one of the following conditions holds:*

- (a)  $f$  has a multiply connected Fatou component;
- (b)  $f$  has growth not exceeding order  $\frac{1}{2}$ , minimal type;
- (c)  $f$  has finite order and Fabry gaps;
- (d)  $f$  exhibits the pits effect (as defined by Littlewood and Offord).

*Results on locally  
connected Julia sets  
of transcendental  
entire functions*

Our final set of results concerns the Julia set  $J(f)$  of a transcendental entire function  $f$  and the property of local connectedness.

The set  $J(f)$  often displays considerable geometric and topological complexity, and it is of interest to ask when it is locally connected at some or all of its points, and what other properties then follow. Rational maps with locally connected Julia sets have been much studied, and several classes of functions are known for which, if the Julia set is connected, then it is also locally connected - see, for example, [56,



Chapter 19] and [25, 53, 86]. Some analogous results have been obtained for transcendental entire functions [20, 58], but the situation is less well understood. More details are given in Section 5.5.

In this thesis, we explore a surprising link between the local connectedness of  $J(f)$  and the spider's web form of  $J(f)$  in the case that  $f$  is a transcendental entire function. Our main result is the following.

**THEOREM 1.15.** *Let  $f$  be a transcendental entire function such that  $J(f)$  is locally connected. Then  $J(f)$  is a spider's web.*

*Locally connected  
Julia sets are  
spider's webs*

In fact, we can show that  $J(f)$  is a spider's web under weaker hypotheses than the local connectedness of  $J(f)$ . Details are given in Section 5.2.

An immediate corollary of Theorem 1.15 is the following.

**COROLLARY 1.16.** *Let  $f$  be a transcendental entire function with an unbounded Fatou component. Then  $J(f)$  is not locally connected.*

In [8, Theorem E], Baker and Domínguez showed that, if a transcendental entire function  $f$  has an unbounded *invariant* Fatou component  $U$ , then  $J(f)$  is not locally connected at any point, except perhaps in the case when  $U$  is a Baker domain and  $f|_U$  is univalent. In this exceptional case it is possible for the boundary of  $U$  to be a Jordan arc [10], but Corollary 1.16 shows that, even then,  $J(f)$  cannot be locally connected at all of its points.

In the opposite direction to Theorem 1.15, we prove the following result.

**THEOREM 1.17.** *Let  $f$  be a transcendental entire function such that  $J(f)$  is a spider's web. Then there exists a subset of  $J(f)$  which is dense in  $J(f)$  and consists of points  $z$  with the property that every neighbourhood of  $z$  contains a continuum in  $J(f)$  that surrounds  $z$ . Each such point  $z$  is a buried point of  $J(f)$  at which  $J(f)$  is locally connected.*

*Spider's web Julia  
sets are locally  
connected at a dense  
subset of points*

It follows from Theorem 1.17 that  $J_+(f)$  is never empty for a transcendental entire function  $f$  for which  $J(f)$  is a spider's web.

Using Theorem 1.17 and a topological result due to Whyburn (see Lemma 5.2), we can build on Theorem 1.15 to obtain more detailed

properties of locally connected Julia sets for transcendental entire functions.

*Properties of locally  
connected Julia sets  
of transcendental  
entire functions*

**THEOREM 1.18.** *Let  $f$  be a transcendental entire function such that  $J(f)$  is locally connected. Then*

- (a)  $J_r(f) \neq \emptyset$ , and every neighbourhood of a point  $z \in J_r(f)$  contains a Jordan curve in  $J(f)$  surrounding  $z$ ;
- (b)  $J(f)$  is a spider's web, and there exists a sequence  $(G_n)_{n \in \mathbb{N}}$  of bounded, simply connected domains having the properties in Definition 1.1 with  $E = J(f)$ , such that the loops  $(\partial G_n)_{n \in \mathbb{N}}$  are Jordan curves.

*Organisation of the  
remaining chapters*

The organisation of the remaining chapters of this thesis is as follows.

Chapter 2 contains background material that we use in later chapters. In Section 2.1, we establish our notation and summarise some standard terminology and results from complex dynamics. Section 2.2 is devoted to a survey of relevant results on the escaping set of a transcendental entire function. Then, in Sections 2.3 to 2.5, we give a brief account of a number of techniques from complex analysis that we use in the proofs of some of our results.

Chapter 3 gives the proofs of our results on the structure of spider's web fast escaping sets. Sections 3.1 and 3.2 deal with some preliminaries. In Section 3.3, we prove Theorem 1.2 on the topological properties of the components of  $A(f)^c$  when  $A_R(f)$  is a spider's web. In Section 3.4, we give a characterisation of the orbits of the components of  $A(f)^c$  and prove Theorem 1.3, whilst in Section 3.5 we restrict our attention to those components of  $A(f)^c$  whose orbits are bounded, proving Theorems 1.5 and 1.6. Our final section, Section 3.6, gives the proof of Theorem 1.7 on singleton periodic components of  $A(f)^c$ .

In Chapter 4 we give proofs and examples of our results on the connectedness properties of the set  $K(f)$  where the iterates of an entire function  $f$  are bounded. In Section 4.1, we prove Theorem 1.11 and Corollary 1.12 on the properties of components of  $K(f)$  for strongly polynomial-like functions. Section 4.2 contains the proofs of our results on the number of components of  $K(f)$  (Theorems 1.8 and

1.10). In Section 4.3, we prove Theorems 1.13 and 1.14 on strongly polynomial-like functions. In Section 4.4, we give several examples of transcendental entire functions for which  $K(f)$  is totally disconnected. Finally, in Section 4.5, we use quasiconformal surgery to construct a transcendental entire function for which  $K(f)$  has a component with empty interior which is not a singleton.

Chapter 5 includes proofs and examples of our results on spiders' webs and locally connected Julia sets of transcendental entire functions. Section 5.1 includes some preliminaries. In Section 5.2, we prove Theorem 1.15 and related results, whilst in Section 5.3 we prove Theorems 1.17 and 1.18. In Section 5.4, we give some results on the residual Julia set  $J_r(f)$ , including the fact that there are classes of functions for which  $J_r(f)$  is itself a spider's web. Finally, in Section 5.5, we give a number of examples to illustrate the results of previous sections. We show that the Julia set of the function  $\sin z$  is a spider's web, and give some new examples of transcendental entire functions for which the Julia set is locally connected.

Our final chapter, Chapter 6, discusses a number of possible directions for future research on the topics covered in the thesis.

## PRELIMINARIES

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In this chapter, we establish our notation and summarise some standard terminology and results from one-dimensional complex dynamics (Section 2.1). In particular, we survey some relevant properties of the escaping set of a transcendental entire function (Section 2.2). Finally, in Sections 2.3 to 2.5, we give brief accounts of a number of techniques from complex analysis that we will need in some of our proofs.

### 2.1 NOTATION AND BACKGROUND ON COMPLEX DYNAMICS

First, we establish some general notational conventions and terminology.

We use the usual notations  $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$  and  $\mathbb{R}$  for the natural numbers, the integers, the rational numbers and the real numbers respectively. We denote the complex plane by  $\mathbb{C}$ , and the extended complex plane or Riemann sphere by  $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ . By  $B(a, r)$ , we mean the open disc  $\{z : |z - a| < r\}$ , by  $\bar{B}(a, r)$  the corresponding closed disc and by  $C(a, r)$  the circle  $\{z : |z - a| = r\}$ . We sometimes use the standard special notation  $\mathbb{D}$  for the open unit disc  $B(0, 1)$ .

*Set theoretic  
notation*

The boundary of a set  $S$  relative to  $\mathbb{C}$  is denoted by  $\partial S$ , the interior of  $S$  by  $\text{int}(S)$  and the closure of  $S$  by  $\bar{S}$ . The *derived set* of  $S$  (i.e. the set of all limit points of  $S$ ) is denoted by  $S'$ . The complement of  $S$  (either in  $\mathbb{C}$  or in  $\hat{\mathbb{C}}$  depending on the context) is often written  $S^c$ , and we call a component of  $S^c$  a *complementary component* of  $S$ .

If  $S$  is a subset of  $\mathbb{C}$ , we use the notation  $\tilde{S}$  to denote the union of  $S$  and all its bounded complementary components (if any). As in [73], we say that  $S$  *surrounds* a set or a point if that set or point lies in a

bounded complementary component of  $S$ . If  $S$  is a bounded domain and  $f$  is an entire function, then we have

$$f(\widetilde{S}) \subset \widetilde{f(S)}, \quad (2.1)$$

since if  $\gamma$  is any Jordan curve in  $S$ , then the image under  $f$  of the inside of  $\gamma$  lies inside  $f(\gamma)$ , and so in  $\widetilde{f(S)}$ .

*Proper maps*

If  $f : G \rightarrow G'$  is an analytic mapping between the domains  $G$  and  $G'$  such that each point in  $f(G)$  has exactly  $k$  preimages in  $G$  (counted with multiplicity) for some fixed  $k \in \mathbb{N}$ , then we call  $f$  a *proper map* and  $k$  its *topological degree* (see [84, Chapter 1]). A proper map  $f : G \rightarrow G'$  always maps boundary points to boundary points in the sense that, for any sequence  $(z_n)_{n \in \mathbb{N}}$  in  $G$  tending to some point in  $\partial G$ , the image sequence  $(f(z_n))$  has all its limit points on  $\partial G'$ .

*The maximum and  
minimum modulus  
functions*

If  $f$  is an entire function, we use the notation

$$M(r, f) := \max_{|z|=r} |f(z)|, \text{ for } r > 0$$

for the *maximum modulus function* of  $f$ , and similarly we denote the *minimum modulus function* by

$$m(r, f) := \min_{|z|=r} |f(z)|, \text{ for } r > 0.$$

We write  $M^n(r, f)$  for the  $n$ th iterate of the maximum modulus function  $M$  with respect to  $r$ , and we sometimes abbreviate  $M(r, f)$  to  $M(r)$  where  $f$  is clear from the context. Note that, for a transcendental entire function  $f$ , it is always possible to choose  $R > 0$  such that  $M(r) > r$  for all  $r \geq R$ .

*Background on  
complex dynamics*

We now summarise some ideas and terminology from one dimensional complex dynamics that are used throughout this thesis. Some of this terminology has already been introduced in Chapter 1 but we repeat it here for convenience. The definitions we give apply to entire functions, but the similar definitions which apply to rational or meromorphic functions may readily be found in the literature (see [14, 24, 56] for rational functions and [15, 59] for transcendental functions).

Let  $f$  be a non-linear entire function and let  $f^n$ ,  $n = 0, 1, 2, \dots$ , denote the  $n$ th iterate of  $f$ . For any  $z \in \mathbb{C}$ , we call the sequence  $(f^n(z))_{n \geq 0}$  the *orbit* of  $z$  under  $f$ .

We say that a point  $z \in \mathbb{C}$  is *periodic* if there is some  $n \in \mathbb{N}$  such that  $f^n(z) = z$ , and the smallest value of  $n$  with this property is called the *period* of  $z$ . A point of period 1 is called a *fixed point*. The nature of a periodic point  $z$  is determined by the value of its *multiplier*  $\lambda = (f^n)'(z)$ . A periodic point is called *attracting* if  $0 \leq |\lambda| < 1$ , *superattracting* if  $\lambda = 0$  and *repelling* if  $|\lambda| > 1$ . If  $|\lambda| = 1$ , so that  $\lambda = e^{2\pi i \theta}$  for some  $\theta \in [0, 1)$ , the periodic point is called *rationally indifferent* or *parabolic* if  $\theta \in \mathbb{Q}$  and *irrationally indifferent* if  $\theta \in \mathbb{R} \setminus \mathbb{Q}$ .

The *Fatou set*  $F(f)$  is the set of points  $z \in \mathbb{C}$  such that the family of functions  $\{f^n : n \in \mathbb{N}\}$  is normal in some neighbourhood of  $z$ , and the *Julia set*  $J(f)$  is the complement of  $F(f)$ . Here, a family  $\mathcal{F}$  of functions analytic on a domain  $G \subset \mathbb{C}$  is *normal* in  $G$  if every sequence of functions  $(f_n) \subset \mathcal{F}$  contains a subsequence which converges locally uniformly on  $G$ . The limit function (which need not be in  $\mathcal{F}$ ) is either analytic or identically equal to infinity. Equivalently,  $\mathcal{F}$  is normal in  $G$  if  $\mathcal{F}$  is equicontinuous in  $G$ .

Evidently,  $F(f)$  is an open set and  $J(f)$  is closed. In fact,  $J(f)$  is perfect (i.e. it contains no isolated points), and either  $J(f) = \mathbb{C}$  or  $J(f)$  has empty interior. For any integer  $n \geq 2$ , we have  $F(f^n) = F(f)$  and  $J(f^n) = J(f)$ .

Two other sets which are important in complex dynamics are the *escaping set* and the set of points whose orbits are bounded under iteration (this set has no commonly accepted name except in polynomial dynamics, where it is called the *filled Julia set*). The *escaping set*  $I(f) = \{z \in \mathbb{C} : f^n(z) \rightarrow \infty \text{ as } n \rightarrow \infty\}$  is the set of points whose orbits tend to infinity. We give further information on  $I(f)$  and related sets in Section 2.2 below. By analogy with the usual symbol for the filled Julia set of a polynomial, we here denote the set of points whose orbits are bounded for a transcendental entire function  $f$  by  $K(f)$ , so that  $K(f) = \{z \in \mathbb{C} : (f^n(z))_{n \in \mathbb{N}} \text{ is bounded}\}$ .

*The Fatou set and  
the Julia set*

*The escaping set and  
the set of points  
whose orbits are  
bounded*

If we say that the set  $S$  is *completely invariant* under a function  $f$ , we mean that  $z \in S$  if and only if  $f(z) \in S$ . Each of the sets  $J(f)$ ,  $F(f)$ ,  $I(f)$  and  $K(f)$  is completely invariant.

Components of the  
Fatou set

A component of the Fatou set  $F(f)$  is often referred to as a *Fatou component*. If  $U = U_0$  is a Fatou component, then for each  $n \in \mathbb{N}$ ,  $f^n(U) \subset U_n$  for some Fatou component  $U_n$ . If  $U = U_n$  for some  $n \in \mathbb{N}$ , we say that  $U$  is *periodic* or *cyclic*, and if  $n = 1$ , that  $U$  is *invariant*; otherwise, we say that it is *aperiodic*. If  $U$  is not eventually periodic, i.e. if  $U_m \neq U_n$  for all  $n > m \geq 0$ , then  $U$  is called a *wandering* Fatou component or a *wandering domain*. Wandering domains can occur for transcendental entire functions but not for polynomials [6, 85].

If  $z$  is an attracting periodic point, the set  $\{z, f(z), \dots, f^{n-1}(z)\}$  where  $n$  is the period of  $z$  is called an *attracting periodic cycle*, and the set of points whose iterates converge to a point in the cycle is the *attracting basin* of the cycle (the *immediate basin* of a point in the cycle is the component of the attracting basin containing that point). *Parabolic* periodic cycles and basins are defined similarly.

The properties of periodic Fatou components are well known, and we summarise the possible types of such components in the following theorem (see, for example, [15, Theorem 6]).

**THEOREM 2.1.** *Let  $f$  be an entire function, and let  $U$  be a periodic component of  $F(f)$  of period  $p$ . Then one of the following holds.*

- $U$  contains an attracting periodic point  $z_0$  of period  $p$ , in which case  $f^{np}(z) \rightarrow z_0$  as  $n \rightarrow \infty$  for all  $z \in U$ , and  $U$  is the immediate attracting basin of  $z_0$ .
- $\partial U$  contains a parabolic periodic point  $z_0$  of period  $p$ , in which case  $f^{np}(z) \rightarrow z_0$  as  $n \rightarrow \infty$  for all  $z \in U$ , and  $U$  is the immediate parabolic basin of  $z_0$ .
- $U$  is analytically conjugate to a Euclidean rotation of the unit disc onto itself, i.e. for some  $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ , there is an analytic homeomorphism  $\phi : U \rightarrow \mathbb{D}$  such that  $\phi(f^p(\phi^{-1}(z))) = e^{2\pi i \alpha} z$ . In this case,  $U$  is called a *Siegel disc*.

- $f^{np}(z) \rightarrow \infty$  as  $n \rightarrow \infty$  for all  $z \in U$ , in which case  $U$  is called a *Baker domain*.

If  $f$  is a polynomial (or indeed any other rational map), Baker domains cannot occur.

If  $U$  is a Fatou component such that  $f^{-1}(U) \subset U$ , then it follows that  $f(U) \subset U$ , and  $U$  is referred to as *completely invariant*. It is shown in [5] that, if  $f$  is a transcendental entire function, there can be at most one such component.

If  $K$  is a component of  $K(f)$ , we call the sequence of components  $K_n$  such that  $f^n(K) \subset K_n$  the *orbit* of  $K$ . Periodic, aperiodic and wandering components of  $K(f)$  are defined as for components of  $F(f)$ . Periodic components of  $K(f)$  always exist and wandering components may exist, both for polynomials (since at most countably many components of  $J(f)$  are eventually periodic - see, for example, [52]) and for transcendental entire functions (see, for example, Theorem 1.3).

*Components of  $K(f)$*

For  $z \in \mathbb{C}$ , the *backwards orbit*  $O^-(z)$  of  $z$  is the set of all preimages of  $z$  under iteration by  $f$ :

$$O^-(z) = \{w \in \mathbb{C} : f^n(w) = z \text{ for some } n \geq 0\}.$$

The *exceptional set*  $E(f)$  is the set of points with a finite backwards orbit under  $f$ . For a transcendental entire function,  $E(f)$  contains at most one point.

In the following lemma, we collect together various well-known properties of the Julia set  $J(f)$ .

*Properties of the  
Julia set*

**LEMMA 2.2.** *Let  $f$  be an entire function and  $J(f)$  be the Julia set of  $f$ . Then:*

- $J(f)$  is compact if  $f$  is a polynomial but unbounded if  $f$  is transcendental;*
- $J(f)$  is the closure of the set of repelling periodic points of  $f$ ;*
- for  $z \in \mathbb{C} \setminus E(f)$  we have  $J(f) = O^-(z)'$ ;*
- if  $K \subset \mathbb{C} \setminus E(f)$  is a compact set and  $G$  is an open neighbourhood of  $z \in J(f)$ , then there exists  $N \in \mathbb{N}$  such that  $f^n(G) \supset K$ , for all  $n \geq N$  (we refer to this property as the 'blowing up' property of  $J(f)$ ).*



A point  $z \in J(f)$  is called a *buried point* if  $z$  does not lie on the boundary of any Fatou component. A *buried component* is a component of  $J(f)$  consisting entirely of buried points. The set of all buried points is called the *residual Julia set* and is denoted by  $J_r(f)$  (see [9, 29] for the properties of this set).

*Singular values*

The dynamical behaviour of an entire function  $f$  is much affected by its set of critical values and finite asymptotic values. If  $f'(z) = 0$  we say that  $z$  is a *critical point* and  $f(z)$  is a *critical value* of  $f$ . A point  $a \in \mathbb{C}$  is a *finite asymptotic value* of  $f$  if there is a curve  $\gamma : [0, \infty) \rightarrow \mathbb{C}$  with  $\gamma(t) \rightarrow \infty$  and  $f(\gamma(t)) \rightarrow a$  as  $t \rightarrow \infty$ . Finite asymptotic values can occur for transcendental entire functions but not for polynomials.

Collectively, the set of critical values and finite asymptotic values of an entire function  $f$  is called its set of *singular values* and is denoted by  $\text{sing}(f^{-1})$ . This set has a close relationship to the periodic cycles of Fatou components of  $f$ . Define the *post-singular set* to be the closure of the set of orbits of the singular values of  $f$ ,

$$P(f) = \overline{\bigcup_{n \in \mathbb{N}} f^n(\text{sing}(f^{-1}))}. \quad (2.2)$$

Then the following holds (see, for example, [15, Theorem 7]):

**THEOREM 2.3.** *Let  $f$  be an entire function, and let  $C = \{U_0, U_1, \dots, U_{p-1}\}$  be a cycle of components of  $F(f)$  of period  $p \geq 1$ .*

- *If  $C$  is a cycle of immediate attracting basins or immediate parabolic basins, then  $U_j \cap \text{sing}(f^{-1}) \neq \emptyset$  for some  $j \in \mathbb{Z}$  with  $0 \leq j \leq p-1$ .*
- *If  $C$  is a cycle of Siegel discs, then  $\partial U_j \subset P(f)$  for all  $j \in \mathbb{Z}$  with  $0 \leq j \leq p-1$ .*

*The class  $\mathcal{S}$  and  
the class  $\mathcal{B}$*

Because of the importance of the set of singular values, it is convenient to have some terminology for functions where this set has particular properties. We say that the transcendental entire function  $f$  is in the *Speiser class*  $\mathcal{S}$  if  $\text{sing}(f^{-1})$  is a finite set, and in the *Eremenko-Lyubich class*  $\mathcal{B}$  if  $\text{sing}(f^{-1})$  is bounded.

The dynamical properties of transcendental entire functions in the classes  $\mathcal{S}$  and  $\mathcal{B}$  were studied by Eremenko and Lyubich in [35]. For example, for  $f \in \mathcal{S}$  they proved that  $f$  has no wandering domains, and

for  $f \in \mathcal{B}$  that  $I(f) \subset J(f)$  so that, in particular,  $f$  has no Baker domains. In many ways, transcendental entire functions in these classes (which include  $\lambda \exp z$  and  $a \cos z + b$  for  $\lambda, a, b \in \mathbb{C}$ ) are the simplest to study dynamically, a fact which has made them a major focus of subsequent research.

## 2.2 THE ESCAPING SET

In this section, we give some background on the properties of the escaping set  $I(f) = \{z \in \mathbb{C} : f^n(z) \rightarrow \infty \text{ as } n \rightarrow \infty\}$  of a transcendental entire function  $f$ .

Although Fatou [37] made some interesting observations about the escaping sets of particular functions (see below), the first results on the properties of  $I(f)$  for a general transcendental entire function are due to Eremenko [33], who proved the following in 1989.

*Properties of the escaping set of a transcendental entire function*

**THEOREM 2.4.** *Let  $f$  be a transcendental entire function and  $I(f)$  be the escaping set of  $f$ . Then*

- (a)  $I(f) \neq \emptyset$ ;
- (b)  $I(f) \cap J(f) \neq \emptyset$ ;
- (c)  $J(f) = \partial I(f)$ , and
- (d) every component of  $\overline{I(f)}$  is unbounded.

In addition Eremenko *conjectured* in [33] that every component of  $I(f)$  is unbounded for any transcendental entire function  $f$  (in which case we could replace  $\overline{I(f)}$  by  $I(f)$  in Theorem 2.4(d)). This is now known as the *Eremenko's conjecture* or, sometimes, as the *weak form* of the conjecture in contrast with a strong form described in the next paragraph.

*Eremenko's conjecture*

In his study of the dynamics of transcendental entire functions, Fatou [37] observed that, for the functions  $f(z) = z + 1 + e^{-z}$  and  $f(z) = c \sin z$  (where  $0 < c < 1$ ), there are infinitely many unbounded curves  $\gamma$  such that  $f^n(z) \rightarrow \infty$  as  $n \rightarrow \infty$  for  $z \in \gamma$ . Fatou also asked whether this property might hold more generally. In [33], Eremenko

made Fatou's question more precise and conjectured that, for a transcendental entire function  $f$ , any point  $z \in I(f)$  can be joined to  $\infty$  by a curve in  $I(f)$ . This statement is sometimes called the *strong form of Eremenko's conjecture*. Clearly if this strong form of the conjecture holds, then so does the weak form.

The identity  $J(f) = \partial I(f)$  in Theorem 2.4(c) explains much of the interest in the escaping set in complex dynamics, namely that its study leads to insights into the properties of  $J(f)$ . Because of its simple definition,  $I(f)$  is sometimes easier to work with than  $J(f)$ . This, together with the two forms of Eremenko's conjecture, have motivated much subsequent research on the escaping sets of transcendental entire functions.

Before discussing this further, we recall some elementary properties of  $I(f)$  for an entire function  $f$ . If  $f$  is a non-linear polynomial, then  $I(f)$  is the basin of attraction of the point at infinity, so that  $I(f) \subset F(f)$  (in contrast with Theorem 2.4(b)) and  $J(f) = \partial I(f)$  (as in Theorem 2.4(c)). Moreover:

- if  $f$  is entire, then  $I(f)$  is completely invariant and  $I(f^n) = I(f)$ , for any  $n \in \mathbb{N}$ ;
- if  $f$  is a polynomial then  $I(f)$  is open, but if  $f$  is transcendental entire, then  $I(f)$  is neither open nor closed.

*The form of the escaping set for functions in the classes  $\mathcal{S}$  and  $\mathcal{B}$*

When Eremenko's paper was written, Devaney and Tangerman [27] had already proved the existence of uncountably many curves to infinity in the escaping sets of a large subclass of the class  $\mathcal{S}$ . They called these structures of unbounded curves *Cantor bouquets* and investigated some of their properties (such Cantor bouquets were later given a topological definition and shown to be homeomorphic to one another - see [1, 13]).

Of course, the existence of Cantor bouquets in  $I(f)$  does not mean that the strong form of Eremenko's conjecture holds since there could still be points in  $I(f)$  not lying on a curve to infinity in  $I(f)$ . However, subsequent research has shown that the conjecture does hold for whole families of functions - see, for example, [81] on the exponential family and [79] on the cosine family. More generally, Rottenfusser,

Rückert, Rempe and Schleicher have proved that the strong form of Eremenko's conjecture holds for any finite composition of functions  $f \in \mathcal{B}$  of finite order [78, Theorem 1.2].

On the other hand, the same paper [78, Theorem 1.1] gives a construction of an entire function in the class  $\mathcal{B}$  for which every path-connected component of  $J(f)$  is bounded. Since  $I(f) \subset J(f)$  for  $f \in \mathcal{B}$  [35], the strong form of Eremenko's conjecture fails for this function. However, Rempe has shown in [71] that, if  $f \in \mathcal{B}$  and the post-singular set  $P(f)$  is bounded, then every component of  $I(f)$  is unbounded, so the weak form of Eremenko's conjecture holds. Since  $P(f)$  is bounded for the function in [78, Theorem 1.1], it follows that this function satisfies the weak form of Eremenko's conjecture but not the strong form.

To summarise, therefore, the strong form of Eremenko's conjecture has been shown to be false in general. By contrast, the weak form of the conjecture remains open for general transcendental entire functions, though a partial result has been obtained by Rippon and Stallard, as we now describe.

Unlike much recent research on the escaping set which has focussed on the class  $\mathcal{B}$ , Rippon and Stallard [71, 73] took as the starting point for their work a subset of the escaping set called the *fast escaping set*, rather than a particular class of functions. First introduced by Bergweiler and Hinkkanen in [19], the fast escaping set  $A(f)$  for a transcendental entire function  $f$  is defined in [73] as follows:

*Properties of the fast escaping set*

$$A(f) = \{z \in \mathbb{C} : \exists \ell \in \mathbb{N} \text{ such that } |f^{n+\ell}(z)| \geq M^n(R, f), \text{ for } n \in \mathbb{N}\},$$

where  $R > 0$  is chosen so that  $M(r, f) > r$  for  $r \geq R$ .

It turns out that  $A(f)$  shares many properties with  $I(f)$ .

**THEOREM 2.5.** [19, 71] *Let  $f$  be a transcendental entire function and let  $A(f)$  be the fast escaping set of  $f$ . Then*

- (a)  $A(f) \neq \emptyset$ ;
- (b)  $A(f) \cap J(f) \neq \emptyset$ ;
- (c)  $J(f) = \partial A(f)$ ;
- (d)  $A(f)$  is completely invariant, and  $A(f^n) = A(f)$  for all  $n \in \mathbb{N}$ ;

(e)  $A(f)$  is neither open nor closed.

In fact, Rippon and Stallard have shown that  $A(f)$  has many stronger properties than  $I(f)$ . We summarise a number of their results in the following theorem.

**THEOREM 2.6.** [73, Theorems 1.1 - 1.3] *Let  $f$  be a transcendental entire function and let  $A(f)$  be the fast escaping set of  $f$ . Then*

- (a) *each component of  $A(f)$  is unbounded;*
- (b) *if  $U$  is a Fatou component of  $f$  that meets  $A(f)$ , then  $\overline{U} \subset A(f)$ ;*
- (c) *if  $f$  has no multiply connected Fatou components, then each component of  $A(f) \cap J(f)$  is unbounded.*

Since  $A(f) \subset I(f)$ , Theorem 2.6(a) implies that at least one component of  $I(f)$  must be unbounded for any transcendental entire function  $f$ , and is therefore a partial result in the direction of the weak form of Eremenko's conjecture.

*Spider's web fast  
escaping sets and  
their properties*

We now recall the basic properties of multiply connected Fatou components for a transcendental entire function, proved by Baker.

**LEMMA 2.7.** [6, Theorem 3.1] *Let  $f$  be a transcendental entire function and let  $U$  be a multiply connected Fatou component. Then*

- $f^n(U)$  is bounded for any  $n \in \mathbb{N}$ ,
- $f^{n+1}(U)$  surrounds  $f^n(U)$  for large  $n$ , and
- $\text{dist}(0, f^n(U)) \rightarrow \infty$  as  $n \rightarrow \infty$ .

In [71, Theorem 2], Rippon and Stallard showed that, for a transcendental entire function  $f$  with a multiply connected Fatou component, both  $A(f)$  and  $I(f)$  are connected and contain the closures of all such Fatou components. Comparing this with Lemma 2.7 leads to the conclusion that here the escaping set takes a novel form, in contrast with the Cantor bouquet structure described above. In [73], Rippon and Stallard called this form of the escaping set a *spider's web* (see Definition 1.1).

The paper [73] is a comprehensive study of the set  $A(f)$  for a transcendental entire function  $f$  and includes many new results on its

properties. Here, Rippon and Stallard introduced the concept of the *levels* of the fast escaping set (see Section 3.1 for more details) and in particular defined the set

$$A_R(f) = \{z \in \mathbb{C} : |f^n(z)| \geq M^n(R, f), \text{ for } n \in \mathbb{N}\}.$$

They showed that whenever  $A_R(f)^c$  has a bounded component, each of the sets  $A_R(f)$ ,  $A(f)$  and  $I(f)$  is a spider's web [73, Theorem 1.4], and proved that this is the case for many classes of functions (not just those with a multiply connected Fatou component) - see [73, Theorem 1.9], and also [55, 83]. Evidently, for all of these functions, the weak form of Eremenko's conjecture holds, since  $I(f)$  is both connected and unbounded.

Under the additional condition that  $A_R(f)$  is a spider's web, further strong properties of  $A(f)$  hold. For example, Rippon and Stallard have proved the following.

**THEOREM 2.8.** [73, Theorems 1.5, 1.6] *Let  $f$  be a transcendental entire function, let  $R > 0$  be such that  $M(r, f) > r$  for  $r \geq R$  and let  $A_R(f)$  be a spider's web.*

- (a) *If  $f$  has no multiply connected Fatou components, then each of the sets  $A_R(f) \cap J(f)$ ,  $A(f) \cap J(f)$ ,  $I(f) \cap J(f)$  and  $J(f)$  is a spider's web.*
- (b) *The function  $f$  has no unbounded Fatou components.*
- (c) *All the components of  $A(f)^c$  are compact.*
- (d) *Every point of  $J(f)$  is the limit of a sequence of points, each of which lies in a distinct component of  $A(f)^c$ .*

We remark in passing that it is not necessary for  $A_R(f)$  to be a spider's web for the escaping set of a transcendental entire function to be connected. For example, although  $I(f)$  is disconnected for many functions in the class  $\mathcal{B}$  (see, for example, [54, Corollary 1.4]), there are also functions in this class for which  $I(f)$  is connected [43, 67, 68] (note that  $f \notin \mathcal{B}$  if  $A_R(f)$  is a spider's web by [73, Theorem 1.8(a)]). Furthermore, Rippon and Stallard have recently given details [75] of

*Other connected  
escaping sets for  
transcendental  
entire functions*

a transcendental entire function for which  $I(f)$  is a spider's web but  $A_{\mathbb{R}}(f)$  is not, answering a question posed in [73].

In Chapter 3 we extend Rippon and Stallard's results in Theorem 2.8 by investigating the structure of the  $A(f)$  spider's web when  $A_{\mathbb{R}}(f)$  is a spider's web, in particular proving some further properties of the set  $A(f)^c$  under this condition (Theorems 1.2 to 1.7).

### 2.3 QUASICONFORMAL MAPPINGS AND SURGERY

In this section we briefly introduce the idea of quasiconformal mappings and the technique of quasiconformal surgery. We limit our discussion to the context of the extended complex plane  $\widehat{\mathbb{C}}$ , and we cover only what is needed for later chapters.

*Quasiconformal  
mappings and their  
properties*

Loosely speaking, a mapping is quasiconformal if it transforms infinitesimal circles to infinitesimal ellipses with bounded eccentricity (as measured by the ratio of the major to the minor axes of the ellipses). For a brief account of quasiconformal mappings see [24, 40], and for a full discussion see [2, 49].

A definition of a quasiconformal mapping can be based on this measure of eccentricity.

**DEFINITION 2.9.** [2, p. 87] Let  $G$  and  $G'$  be domains in  $\widehat{\mathbb{C}}$ , and let  $f$  be an orientation-preserving homeomorphism of  $G$  onto  $G'$ . The *circular dilatation* of  $f$  at the point  $z_0 \in G$  is given by

$$H_f(z_0) = \limsup_{r \rightarrow 0^+} \frac{\max_{|z-z_0|=r} |f(z) - f(z_0)|}{\min_{|z-z_0|=r} |f(z) - f(z_0)|}.$$

Then  $f : G \rightarrow G'$  is *quasiconformal* if its circular dilatation function  $H_f$  has a finite upper bound in  $G$ , and  $f$  is *K-quasiconformal* (where  $1 \leq K < \infty$ ) if  $H_f(z) \leq K$  for almost every  $z \in G$ .

If  $f$  is differentiable at the point  $z_0 \in G$ , we also define the *complex dilatation* of  $f$ ,

$$\mu_f(z_0) = \frac{f_{\bar{z}}(z_0)}{f_z(z_0)},$$

which gives the direction of maximal distortion as well as a measure of eccentricity. The condition  $H_f(z) \leq K$  is equivalent to

$$|\mu_f(z)| \leq (K - 1)/(K + 1).$$

Much of the usefulness of quasiconformal mappings in complex dynamics derives from a remarkable result known as the measurable Riemann mapping theorem (see, for example, [49, p. 194]). This states that, if  $G$  and  $G'$  are conformally equivalent simply connected domains and  $\mu$  is a measurable function in  $G$  with  $\sup_{z \in G} |\mu(z)| < 1$ , then there exists a quasiconformal mapping  $f : G \rightarrow G'$  whose complex dilatation coincides with  $\mu$  almost everywhere, and this mapping is uniquely determined up to a conformal mapping of  $G'$  onto itself.

*The measurable  
Riemann mapping  
theorem*

Using this result, it is possible to use various techniques to construct analytic mappings with specified dynamical properties (see the article by Shishikura in [2]). As an example, Sullivan [85] used a technique known as quasiconformal deformation in his proof that rational maps have no wandering domains. The measurable Riemann mapping theorem is also the basis for *quasiconformal surgery*, whereby two different analytic functions are 'glued' together via conjugation with a quasiconformal mapping to produce another analytic function which combines the dynamical properties of both original functions.

*Quasiconformal  
surgery*

The method of quasiconformal surgery for rational maps was pioneered by Douady and Hubbard [32] and by Shishikura [82] in the 1980s. Douady and Hubbard used the method in developing their theory of polynomial-like mappings (see Section 2.4), whilst Shishikura used it to derive a sharp bound on the number of periodic cycles of Fatou components for a rational map.

In [47], Kisaka and Shishikura modified the methodology in [82] to apply to entire functions. The key result, which we use in Section 4.5, is the following theorem. Here, a mapping  $g : \mathbb{C} \rightarrow \mathbb{C}$  is *K-quasiregular* if it can be written as  $g = f \circ \phi$ , where  $\phi$  is *K-quasiconformal* and  $f$  is entire (see [69] for further information on quasiregular mappings).



**THEOREM 2.10.** [47, Theorem 3.1] *Let  $g : \mathbb{C} \rightarrow \mathbb{C}$  be a quasiregular mapping. Suppose that there are disjoint measurable sets  $E_j \subset \mathbb{C}, j \in \mathbb{N}$ , such that:*

- (a) *for almost every  $z \in \mathbb{C}$ , the  $g$ -orbit of  $z$  meets  $E_j$  at most once for every  $j$ ;*
- (b)  *$g$  is  $K_j$ -quasiregular on  $E_j$ ;*
- (c)  $K_\infty := \prod_{j=1}^{\infty} K_j < \infty$ ;
- (d)  *$g$  is analytic almost everywhere outside  $\bigcup_{j=1}^{\infty} E_j$ .*

*Then there exists a  $K_\infty$ -quasiconformal mapping  $\phi : \mathbb{C} \rightarrow \mathbb{C}$  such that  $f = \phi \circ g \circ \phi^{-1}$  is an entire function.*

To use this result to construct an entire function with specified dynamical behaviour, the first step is to construct a quasiregular mapping  $g$  with similar dynamical behaviour by gluing together suitable entire functions by interpolation. Subject to the stated conditions, Theorem 2.10 then guarantees the existence of a quasiconformal mapping  $\phi$  such that  $f = \phi \circ g \circ \phi^{-1}$  is entire, and this conjugation with a quasiconformal mapping ensures that the key features of the dynamical behaviour of  $g$  are passed to the entire function  $f$ .

Using this technique, Kisaka and Shishikura [47] proved a number of results on the connectivity of wandering domains for transcendental entire functions, and Bergweiler [17] demonstrated the existence of a transcendental entire function with both multiply connected and simply connected wandering domains. In Section 4.5, we modify Bergweiler's construction to show that there is a transcendental entire function  $f$  for which the set  $K(f)$  has a component with empty interior that is not a singleton.

## 2.4 POLYNOMIAL-LIKE MAPPINGS

In later chapters, we often use Douady and Hubbard's notion of a polynomial-like mapping [32], and accordingly we make some preliminary remarks about such mappings in this section.

Informally, a polynomial-like mapping of degree  $d$  is a mapping from one topological disc  $V$  to another  $W$ , where  $\bar{V} \subset W$ , such that

each point in  $W$  has exactly  $d$  preimages in  $V$ . Of course, this phenomenon occurs if the mapping is a polynomial restricted to a large enough set, but it also occurs for other functions where the mapping behaves locally like a polynomial. For example [32, p. 295], if  $f(z) = \cos z - 2$  and  $V = \{z \in \mathbb{C} : |\operatorname{Re}(z)| < 2, |\operatorname{Im}(z)| < 3\}$ , then  $f : V \rightarrow f(V)$  is polynomial-like of degree 2.

The formal definition is as follows.

**DEFINITION 2.11.** Let  $V$  and  $W$  be bounded, simply connected domains with smooth boundaries such that  $\bar{V} \subset W$ . Let  $f$  be a proper analytic mapping of  $V$  onto  $W$  with  $d$ -fold covering, where  $d \geq 2$ . Then the triple  $(f; V, W)$  is termed a *polynomial-like mapping* of degree  $d$ .

*Definition of a polynomial-like mapping and its filled Julia set*

The *filled Julia set*  $K(f; V, W)$  of the polynomial-like mapping  $(f; V, W)$  is defined to be the set of all points whose orbits lie entirely in  $V$ , i.e.

$$K(f; V, W) = \bigcap_{k \geq 0} f^{-k}(V).$$

For our purposes, the most useful property of polynomial-like mappings is the fact that each such mapping is quasiconformally conjugate to a polynomial of the same degree. This is the substance of Douady and Hubbard's Straightening Theorem, whose proof relies on the measurable Riemann mapping theorem quoted in Section 2.3.

**THEOREM 2.12.** [32, Theorem 1] If  $(f; V, W)$  is a polynomial-like mapping of degree  $d \geq 2$ , then there exists a quasiconformal mapping  $\phi : \mathbb{C} \rightarrow \mathbb{C}$  and a polynomial  $g$  of degree  $d$  such that  $\phi \circ f = g \circ \phi$  on  $\bar{V}$ . Moreover

*The Straightening Theorem for polynomial-like mappings*

$$\phi(K(f; V, W)) = K(g),$$

where  $K(g)$  is the filled Julia set of the polynomial  $g$ .

This theorem helps to explain why it is that distorted copies of polynomial Julia sets appear in the dynamical planes of many types of functions. Indeed, many aspects of the dynamics of the polynomial  $g$  are preserved by the quasiconformal mapping, including the existence of critical points and the nature of Fatou components. In

the context of the present study, this enables us to follow Kisaka [46], Zheng [91] and Eremeko and Lyubich [36] in using results from polynomial dynamics to obtain new results for transcendental entire functions - see especially Sections 3.5 and 4.1.

## 2.5 THE AHLFORS FIVE ISLANDS THEOREM

Finally in this chapter, we introduce the Ahlfors five islands theorem, a version of which we use in Sections 3.6 and 5.3.

The five islands theorem is part of the theory of covering surfaces for which Ahlfors was awarded one of the first two Fields medals in 1936. This theory is a geometric counterpart to Nevanlinna's value distribution theory for meromorphic functions. A self-contained account of the theory is given in Hayman [39].

There are several forms of the five islands theorem. The form we state here is quoted by Bergweiler in [16].

**THEOREM 2.13.** *Let  $D_1, \dots, D_5$  be Jordan domains on the Riemann sphere with pairwise disjoint closures. Let  $D \subset \hat{\mathbb{C}}$  be a domain, and denote by  $\mathcal{F}(D, \{D_j\}_{j=1}^5)$  the family of all meromorphic functions  $f : D \rightarrow \hat{\mathbb{C}}$  with the property that no subdomain of  $D$  is mapped conformally onto one of the domains  $D_j$  by  $f$ . Then  $\mathcal{F}(D, \{D_j\}_{j=1}^5)$  is a normal family.*

The first use of the five islands theorem in complex dynamics was by Baker [4], who used it in his proof that the repelling periodic points of a transcendental entire function  $f$  are dense in  $J(f)$ . As discussed by Bergweiler in [16], the theorem has found a number of applications in complex dynamics since the work of Baker, being used (for example) in proofs that a transcendental entire function has infinitely many repelling periodic points of period  $n$  for every  $n \geq 2$ , and that the Hausdorff dimension of the Julia set of a meromorphic function is strictly greater than 0 (references are given in [16]).

The application that we adapt in this thesis is Domínguez' [28] use of the five islands theorem to show that, if  $f$  is a transcendental entire function with a multiply connected Fatou component, then  $J(f)$  has buried singleton components, and such components are dense in

$J(f)$  (see also [29]). In [16], Bergweiler gives an alternative proof of this result (also based on the five islands theorem), using a method involving the construction of a singleton component of  $J(f)$  which is also a repelling periodic point of  $f$ . We adapt Bergweiler's method in our proofs of Theorems 1.7 and 1.17.

For this purpose, we use the following corollary of the Ahlfors five islands theorem, proved in [16] for a wide class of meromorphic functions, but here stated for transcendental entire functions since this is all we need.

**PROPOSITION 2.14.** *Let  $f$  be a transcendental entire function, and suppose there are bounded Jordan domains  $D_1, D_2, D_3 \subset \mathbb{C}$  with pairwise disjoint closures. Let  $V_1, V_2, V_3$  be domains satisfying  $V_j \cap J(f) \neq \emptyset$  and  $V_j \subset D_j$  for  $j \in \{1, 2, 3\}$ . Then there exist  $\mu \in \{1, 2, 3\}$ ,  $n \in \mathbb{N}$  and a domain  $U \subset V_\mu$  such that  $f^n : U \rightarrow D_\mu$  is conformal.*

## THE STRUCTURE OF SPIDER'S WEB FAST ESCAPING SETS

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Let  $f$  be a transcendental entire function, let  $R > 0$  be such that  $M(r, f) > r$  for  $r \geq R$ , and let  $A_R(f)$  be a spider's web. Recall that

$$A_R(f) = \{z \in \mathbb{C} : |f^n(z)| \geq M^n(R, f), \text{ for } n \in \mathbb{N}\},$$

and that the fast escaping set  $A(f)$  of  $f$  is defined as follows:

$$A(f) = \{z \in \mathbb{C} : \exists \ell \in \mathbb{N} \text{ such that } |f^{n+\ell}(z)| \geq M^n(R, f), \text{ for } n \in \mathbb{N}\}.$$

In this chapter, we prove new results on the components of  $A(f)^c$  when  $A_R(f)$  is a spider's web (Theorems 1.2 to 1.7). The components of  $A(f)^c$  can be thought of as the 'holes' in the  $A(f)$  spider's web. Rippon and Stallard have already shown that, when  $A_R(f)$  is a spider's web, many strong dynamical properties hold (see Theorem 2.8). Our results build on this work and show that, when  $A_R(f)$  is a spider's web, the  $A(f)$  spider's web has a very intricate and dynamically rich structure.

The organisation of the chapter is as follows. Section 3.1 gives some background on the properties of the sets  $A_R(f)$  and  $A(f)$  from the work of Rippon and Stallard, whilst in Section 3.2 we state and prove some preliminary results which will be needed in later sections. In Section 3.3, we prove Theorem 1.2 on the topological properties of the components of  $A(f)^c$  when  $A_R(f)$  is a spider's web. In Section 3.4, we give a characterisation of the orbits of the components of  $A(f)^c$  and prove Theorem 1.3 on the number of components of  $A(f)^c$  with different orbit types. Section 3.5 is devoted to those components of  $A(f)^c$  whose orbits are bounded, and we prove Theorems 1.5 and 1.6. Our final section, Section 3.6, gives the proof of Theorem 1.7 on singleton periodic components of  $A(f)^c$ .

*Organisation of  
the chapter*

3.1 PROPERTIES OF  $A_R(f)$ ,  $A(f)$  AND RELATED SETS

In this section, we summarise a number of basic properties of  $A_R(f)$ ,  $A(f)$  and related sets which we use throughout the chapter and elsewhere in the thesis. These are taken from [73], which should be consulted for full details and proofs.

*The levels of the  
fast escaping set*

One of the most fruitful innovations in [73] is the notion of the *levels* of the fast escaping set. Let  $R > 0$  be such that  $M(r, f) > r$  for all  $r \geq R$ . Then for  $\ell \in \mathbb{Z}$ , the  $\ell$ th level of  $A(f)$  with respect to  $R$  is the set

$$A_R^\ell(f) = \{z \in \mathbb{C} : |f^n(z)| \geq M^{n+\ell}(R, f), \text{ for } n \in \mathbb{N}, n + \ell \in \mathbb{N}\};$$

equivalently, the  $(-\ell)$ th level of  $A(f)$  is

$$A_R^{-\ell}(f) = \{z \in \mathbb{C} : |f^{n+\ell}(z)| \geq M^n(R, f), \text{ for } n \in \mathbb{N}, n + \ell \in \mathbb{N}\}.$$

In particular, note that  $A_R^0(f) = A_R(f)$ . Working with the levels of  $A(f)$  leads both to simplified proofs of results obtained previously, and to deeper insights into the structure of  $A(f)$ .

From the definitions of  $A(f)$  and its levels, we have

$$A(f) = \bigcup_{\ell \in \mathbb{N}} A_R^{-\ell}(f), \quad (3.1)$$

and

$$f(A_R^\ell(f)) \subset A_R^{\ell+1}(f) \subset A_R^\ell(f), \text{ for } \ell \in \mathbb{Z}. \quad (3.2)$$

These relations easily give that  $A(f)$  is completely invariant. Note also that  $A(f)$  is independent of  $R$  - see [73, Theorem 2.2].

Some basic properties of  $A_R(f)$  spiders' webs are given in the following Lemma.

*Spider's webs and  
the levels of  $A(f)$*

**LEMMA 3.1.** [73, Lemma 7.1(a)-(c)] *Let  $f$  be a transcendental entire function, let  $R > 0$  be such that  $M(r, f) > r$  for  $r \geq R$  and let  $\ell \in \mathbb{Z}$ .*

- (a) *If  $G$  is a bounded component of  $A_R^\ell(f)^c$ , then  $\partial G \subset A_R^\ell(f)$  and  $f^n$  is a proper map of  $G$  onto a bounded component of  $A_R^{n+\ell}(f)^c$ , for each  $n \in \mathbb{N}$ .*

- (b) If  $A_R^\ell(f)^c$  has a bounded component, then  $A_R^\ell(f)$  is a spider's web and hence every component of  $A_R^\ell(f)^c$  is bounded.
- (c)  $A_R(f)$  is a spider's web if and only if  $A_R^\ell(f)$  is a spider's web.

Next, we recall the following definition.

**DEFINITION 3.2.** [73, Definition 7.1] Let  $f$  be a transcendental entire function and let  $R > 0$  be such that  $M(r, f) > r$  for  $r \geq R$ . If  $A_R(f)$  is a spider's web then, for each  $n \geq 0$ , let

*Fundamental holes and loops for  $A_R(f)$*

- $H_n$  denote the component of  $A_R^n(f)^c$  containing 0, and
- $L_n$  denote its boundary,  $\partial H_n$ .

We say that  $(H_n)_{n \geq 0}$  is the *sequence of fundamental holes* for  $A_R(f)$  and  $(L_n)_{n \geq 0}$  is the *sequence of fundamental loops* for  $A_R(f)$ . Note that  $L_n$  may have bounded complementary components other than  $H_n$ .

The following lemma gives some properties of these sequences.

**LEMMA 3.3.** [73, Lemma 7.2] Let  $f$  be a transcendental entire function and let  $R > 0$  be such that  $M(r, f) > r$  for  $r \geq R$ . Suppose that  $A_R(f)$  is a spider's web, and that  $(H_n)_{n \geq 0}$  and  $(L_n)_{n \geq 0}$  are respectively the sequences of fundamental holes and loops for  $A_R(f)$ . Then:

*Properties of fundamental holes and loops*

- (a)  $H_n \supset \{z : |z| < M^n(R)\}$  and  $L_n \subset A_R^n(f)$ , for  $n \geq 0$ ;
- (b)  $H_{n+1} \supset H_n$ , for  $n \geq 0$ ;
- (c) for  $n \in \mathbb{N}$  and  $m \geq 0$ ,

$$f^n(H_m) = H_{m+n} \text{ and } f^n(L_m) = L_{m+n};$$

- (d) there exists  $N \in \mathbb{N}$  such that, for  $n \geq N$  and  $m \geq 0$ ,

$$L_{n+m} \cap L_m = \emptyset;$$

- (e) if  $\ell \in \mathbb{Z}$  and  $G$  is a component of  $A_R^\ell(f)^c$ , then  $f^n(G) = H_{n+\ell}$  and  $f^n(\partial G) = L_{n+\ell}$ , for  $n$  sufficiently large;
- (f) if there are no multiply connected Fatou components, then  $L_n \subset J(f)$  for  $n \geq 0$ .

## 3.2 PRELIMINARY RESULTS

We now state and prove a simple topological characterisation of the buried components of a closed set. We need the result only where the closed set is the Julia set of a transcendental entire function, but we present it in a more general form to bring out its essentially topological nature.

Our proof makes use of the following results from plane topology, which we state here for ease of reference (see [60], pages 124 and 143).

LEMMA 3.4. (a) If  $K_1$  and  $K_2$  are two components of a closed set  $F$  in  $\widehat{\mathbb{C}}$ , then there is a Jordan curve in  $F^c$  that separates  $K_1$  and  $K_2$ .

(b) If  $G$  is a domain in  $\widehat{\mathbb{C}}$ , then each component of  $G^c$  contains just one component of  $\partial G$ .

Buried points  
and components  
of a closed set

Recall that, if  $F$  is a closed set in  $\widehat{\mathbb{C}}$ , and  $K$  is a component of  $F$ , then

- $z \in F$  is a *buried point* of  $F$  if  $z$  does not lie on the boundary of any component of  $F^c$ , and
- $K$  is a *buried component* of  $F$  if  $K$  consists entirely of buried points of  $F$ .

In particular, a buried point of  $J(f)$  is a point of  $J(f)$  that does not lie on the boundary of any Fatou component, and a buried component of  $J(f)$  is a component of  $J(f)$  consisting entirely of such buried points.

Topological  
characterisation of  
buried components

THEOREM 3.5. Let  $K$  be a component of a closed set  $F$  in  $\widehat{\mathbb{C}}$ . Then  $K$  is a buried component of  $F$  if and only if, for each component  $L$  of  $K^c$ , and any closed subset  $B$  of  $L$ , there is a component of  $F^c$  that separates  $B$  from  $K$  and whose boundary does not meet  $K$ .

*Proof.* Let  $K$  be a buried component of  $F$ , let  $L$  be a component of  $K^c$  and let  $B$  be any closed subset of  $L$ . Then  $X = B \cup F$  is closed in  $\widehat{\mathbb{C}}$ ,  $K$  is a component of  $X$  and  $B$  lies in some other component of  $X$ , say  $X'$ .

Then it follows from Lemma 3.4(a) that there is a Jordan curve  $C$  separating  $K$  from  $X'$  in such a way that  $C$  lies in  $X^c \subset F^c$ . Since  $C$  is connected, it must lie in some component  $G$  of  $F^c$ . Furthermore, by Lemma 3.4(b), the complementary component of  $G$  containing  $K$



contains exactly one component ( $D$ , say) of  $\partial G$ . Since  $K$  is a buried component of  $F$ , we therefore have  $D \cap K = \emptyset$ , as required.

To prove the converse, let  $K$  be a component of  $F$ . Suppose there exists some component  $G$  of  $F^c$  and some  $z \in K$  such that  $z \in \partial G$ . Let  $L$  be the component of  $K^c$  containing  $G$ , and let  $B$  be a closed subset of  $G$ . Now suppose that there is a component  $G'$  of  $F^c$  separating  $B$  from  $K$  (and hence  $B$  from  $z$ ), whose boundary does not meet  $K$ . Then since  $B \subset G$  and  $z \in \partial G$ ,  $G'$  must meet  $G$ . But  $G$  is a component of  $F^c$ , so this means that  $G' = G$ , which is a contradiction because  $\partial G \cap K \neq \emptyset$ .  $\square$

We will also need the following result on mappings of the components of  $A(f)^c$ .

**THEOREM 3.6.** *Let  $f$  be a transcendental entire function and let  $R > 0$  be such that  $M(r, f) > r$  for  $r \geq R$ . If  $A_R(f)$  is a spider's web, and  $K$  is a component of  $A(f)^c$ , then  $f(K)$  is also a component of  $A(f)^c$ .*

*Components of  $A(f)^c$  are mapped onto other such components*

*Proof.* As  $A(f)$  is completely invariant, it is clear that  $f(K)$  must lie in a component of  $A(f)^c$ , say  $K'$ .

Since  $A_R(f)$  is a spider's web, components of  $A(f)^c$  are compact by Theorem 2.8(c), so each component of  $f^{-1}(K')$  must be closed and lie in some component of  $A(f)^c$ . One such component must contain  $K$ , and indeed be equal to  $K$  since  $K$  is itself a component of  $A(f)^c$ .

Suppose  $w \in K' \setminus f(K)$ . Since  $A_R(f)$  is a spider's web, there exists a bounded, simply connected domain  $G$  containing  $K$  whose boundary lies in  $A(f)$ . The domain  $G$  can contain only a finite number of components of  $f^{-1}(K')$ .

Now by Lemma 3.4(a), there is a Jordan curve  $C$  lying in  $G$  that surrounds  $K$  and separates  $K$  from all other components of  $f^{-1}(K')$ . It follows that  $f(C)$  is a curve that surrounds  $f(K)$  and does not meet  $K'$ . Furthermore,  $f(C)$  cannot surround  $w \in K'$  since  $C$  does not surround any solution of  $f(z) = w$ . This contradicts the connectedness of  $K'$ , and it follows that  $K' \setminus f(K) = \emptyset$ . Thus  $f(K)$  is a component of  $A(f)^c$ , as required.  $\square$

Under the conditions of Theorem 3.6, we call the sequence of iterates of  $K$  its *orbit*, and any infinite subsequence of its iterates a *suborbit*.

If  $f^p(K) = K$  for some  $p \in \mathbb{N}$ , then we say that  $K$  is a *periodic component* of  $A(f)^c$ . If  $f^m(K) \neq f^n(K)$  for all  $m > n \geq 0$ , then we say that  $K$  is a *wandering component* of  $A(f)^c$ .

### 3.3 THE TOPOLOGY OF COMPONENTS OF $A(f)^c$

*Restatement of  
Theorem 1.2*

In this section we prove Theorem 1.2. Let  $f$  be a transcendental entire function, let  $R > 0$  be such that  $M(r, f) > r$  for  $r \geq R$  and let  $A_R(f)$  be a spider's web. Let  $K$  be a component of  $A(f)^c$ . Then Theorem 1.2 states that:

- (a)  $\partial K \subset J(f)$  and  $\text{int}(K) \subset F(f)$ . In particular,  $\overline{U} \subset K$  for every Fatou component  $U$  for which  $K \cap \overline{U} \neq \emptyset$ .
- (b) Every neighbourhood of  $K$  contains a closed subset of  $A(f) \cap J(f)$  surrounding  $K$ . If  $K$  has empty interior, then  $K$  consists of buried points of  $J(f)$ .
- (c) If  $f$  has a multiply connected Fatou component, then every neighbourhood of  $K$  contains a multiply connected Fatou component surrounding  $K$ . If, in addition,  $K$  has empty interior, then  $K$  is a buried component of  $J(f)$ .

*Proof of part (a):  
Components of  
 $A(f)^c$  and the Fatou  
and Julia sets*

*Proof of Theorem 1.2.* By Theorem 2.5(c) we have  $J(f) = \partial A(f)$ , so it is immediate that  $\partial K \subset J(f)$  and  $\text{int}(K) \subset F(f)$ . If  $K$  meets the closure of some Fatou component  $U$ , then  $U \cap A(f) = \emptyset$ , for otherwise  $\overline{U} \subset A(f)$  by Theorem 2.6(b). Hence  $U \subset K$ . But since  $A_R(f)$  is a spider's web,  $K$  is compact by Theorem 2.8(c), so  $\overline{U} \subset K$ . This proves part (a).

For the proof of parts (b) and (c) observe that, using (3.1) and (3.2), we can write

$$K = \bigcap_{\ell \in \mathbb{N}} G_\ell, \quad \text{with } G_\ell \supset G_{\ell+1} \text{ for all } \ell \in \mathbb{N},$$

where  $G_\ell$  is the component of  $A_R^{-\ell}(f)^c$  containing  $K$ . Thus, for any neighbourhood  $V$  of  $K$ , there exists  $M \in \mathbb{N}$  such that  $\overline{G}_\ell \subset V$  for all  $\ell \geq M$ .

As in Definition 3.2, let  $(H_n)_{n \geq 0}$  and  $(L_n)_{n \geq 0}$  be the sequences of fundamental holes and loops for  $A_R(f)$ . Then it follows from Lemma 3.3(e) that, for any  $m \geq 0$  and for sufficiently large  $n \geq m$ :

$$f^{M+n}(\partial G_{M+m}) = L_{n-m}.$$

Now, for any  $n \geq 0$ ,  $L_n \subset A(f)$  and, if  $f$  has no multiply connected Fatou components, we also have  $L_n \subset J(f)$  by Lemma 3.3(f). Since  $L_n$  is closed, it follows that  $V$  contains a closed subset of  $A(f) \cap J(f)$  that surrounds  $K$ . Furthermore, if  $K$  has empty interior, then part (a) implies that  $K$  consists of buried points of  $J(f)$ . This proves part (b) in the case where there are no multiply connected Fatou components.

Now suppose that  $f$  has a multiply connected Fatou component,  $U$ . Note that  $\bar{U} \subset A(f)$ , by [71, Theorem 2]. Thus part (c) of Theorem 1.2 implies part (b), and we need only prove part (c).

Since  $L_0$  is bounded, Lemma 2.7 implies that we can choose  $k \in \mathbb{N}$  so that

$$f^{j+1}(U) \text{ surrounds } f^j(U) \text{ for } j \geq k, \text{ and } f^k(U) \text{ surrounds } L_0.$$

Again, since  $f^k(U)$  is bounded, it follows from Lemma 3.3(a) that we may also choose  $P \in \mathbb{N}$  so that  $f^k(U) \subset H_P$  (see Figure 1). Moreover, Lemma 3.3(e) shows that there exists  $N \in \mathbb{N}$  (depending on  $M$  and  $P$ ) such that the following hold:

$$\begin{aligned} f^{M+N+P}(G_M) &= H_{N+P}, \\ f^{M+N+P}(\partial G_M) &= L_{N+P}, \\ f^{M+N+P}(G_{M+P}) &= H_N, \text{ and} \\ f^{M+N+P}(\partial G_{M+P}) &= L_N. \end{aligned}$$

Since  $f^k(U) \subset H_P$ , it is clear that  $f^{N+k}(U) \subset H_{N+P}$ , and by our choice of  $k$ ,  $f^{N+k}(U)$  surrounds  $f^k(U)$ . We claim that  $f^{N+k}(U)$  also surrounds  $H_N$ . For let  $W$  denote the interior of the complementary

*Proof of part (b):*  
Components of  
 $A(f)^c$  and buried  
points of  $J(f)$

*Proof of part (c):*  
Components of  
 $A(f)^c$  and buried  
components of  $J(f)$

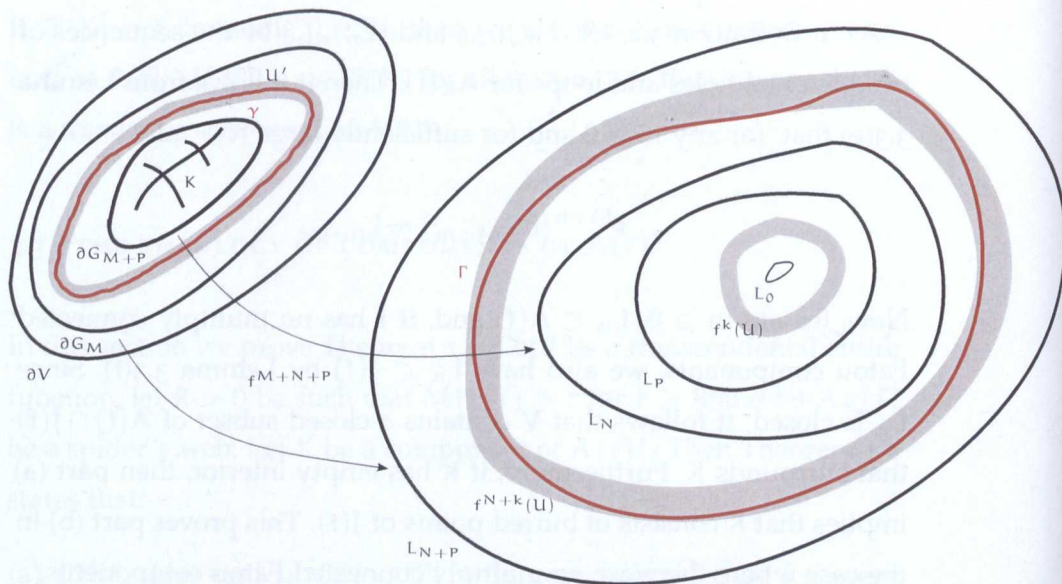


Figure 1: Proof of Theorem 1.2(c). Here  $K$  is shown with no interior, and multiply connected Fatou components are represented by the grey shaded areas. Recall that  $L_n = \partial H_n$  for  $n \in \mathbb{N}$ .

component of  $f^k(U)$  that contains  $H_0$ . Then  $W \subset \widetilde{f^k(U)}$ , so by (2.1) we have

$$H_N = f^N(H_0) \subset f^N(W) \subset f^N(\widetilde{f^k(U)}) \subset \widetilde{f^{N+k}(U)}.$$

Now  $\partial W \subset \partial f^k(U) \subset J(f)$ , and thus it follows that  $\partial f^N(W) \subset f^N(\partial W)$  cannot meet  $f^{N+k}(U)$ . We have therefore shown that  $f^N(W)$  lies in  $\widetilde{f^{N+k}(U)}$ , but that its boundary does not meet  $f^{N+k}(U)$ . Thus  $f^{N+k}(U)$  surrounds  $f^N(W)$  and hence  $H_N$ , as claimed.

We now show that  $G_M$  must contain a multiply connected Fatou component and that this surrounds  $K$ . To do this, let  $\Gamma$  be a Jordan curve in  $f^{N+k}(U)$  that surrounds 0 (see Figure 1). Then, of the finitely many components of  $f^{-(M+N+P)}(\widetilde{\Gamma})$  that lie in  $G_M$ , one must contain  $G_{M+P}$ , since  $f^{M+N+P}(G_{M+P}) = H_N \subset \widetilde{\Gamma}$ . Call this component  $\Lambda$ , and its boundary  $\gamma$ . Since  $f^{M+N+P}$  is a proper map of the interior of  $\Lambda$  onto the interior of  $\widetilde{\Gamma}$ , we have  $f^{M+N+P}(\gamma) = \Gamma$ , and thus  $\gamma$  must lie in a Fatou component,  $U'$  say, that is contained in  $G_M$ . Furthermore,  $U'$  is multiply connected, since  $\gamma$  surrounds  $G_{M+P}$  which contains  $\partial K \subset J(f)$ . Thus  $G_M$  contains a multiply connected Fatou component

surrounding  $K$ , and therefore so does our arbitrary neighbourhood  $V$  of  $K$ .

Finally, suppose that  $K$  has empty interior, so  $K \subset J(f)$ . Since  $A(f)$  is connected,  $K^c$  has only one component, and the remainder of part (c) therefore follows immediately from Theorem 3.5.  $\square$

### 3.4 ORBITS OF COMPONENTS OF $A(f)^c$

In this section, we give a characterisation of the orbits of the components of  $A(f)^c$  when  $A_R(f)$  is a spider's web, and prove that  $A(f)^c$  then has uncountably many components of various types (Theorem 1.3). To this end, we first describe a natural partition of the plane that enables us to encode information about the orbits of the components of  $A(f)^c$ .

Throughout this section, we assume that  $f$  is a transcendental entire function, that  $R > 0$  is such that  $M(r, f) > r$  for  $r \geq R$ , and that  $A_R(f)$  is a spider's web. Recall from Theorem 3.6 that  $f$  maps a component  $K$  of  $A(f)^c$  onto another such component.

To construct the partition, we proceed as follows. Let  $(L_m)_{m \geq 0}$  be the sequence of fundamental loops for  $A_R(f)$ , as in Definition 3.2. Now, by Lemma 3.3(c) and (d), there exists  $N \in \mathbb{N}$  such that, for  $m \geq 0$ ,

$$L_{N+m} \cap L_m = \emptyset \text{ and } f^N(L_{mN}) = L_{(m+1)N}.$$

Thus  $(L_{mN})_{m \geq 0}$  is a sequence of disjoint loops, and  $f^N$  maps any such loop onto its successor in the sequence. We use these loops to define our partition. To simplify the exposition, we assume (without loss of generality) that  $N = 1$ , so that our sequence of disjoint loops is  $(L_m)_{m \geq 0}$ .

Now define

$$B_0 = H_0, \tag{3.3}$$

and

$$B_m = H_m \setminus H_{m-1}, \text{ for } m \geq 1, \tag{3.4}$$

*Constructing the  
partition of the plane*

where  $(H_m)_{m \geq 0}$  is the sequence of fundamental holes for  $A_R(f)$ . Then, for each  $m \geq 1$ ,  $B_m$  is a connected set surrounding 0 and

$$\partial B_m = L_m \cup L_{m-1}.$$

Also, for any  $k \in \mathbb{N}$ ,

$$\bigcup_{m \geq k} B_m = \mathbb{C} \setminus H_{k-1},$$

and indeed  $\bigcup_{m \geq 0} B_m = \mathbb{C}$ . It follows that the sets  $B_m$ ,  $m \geq 0$ , form a partition of the plane.

*Definition of the itinerary of a point*

Hence, for each point  $z \in \mathbb{C}$ , there is a unique sequence of non-negative integers  $\underline{s} = s_0 s_1 s_2 \dots$  (which we call the *itinerary* of  $z$  with respect to  $A_R(f)$ ), such that

$$f^k(z) \in B_{s_k}, \text{ for } k \in \mathbb{N} \cup \{0\}.$$

Evidently, the itinerary of a point encodes information about its orbit, and we now investigate which orbits are possible. We begin with the following lemma, whose proof is based on an argument in the proof of [72, Lemma 6].

*Possible mappings within the partition*

**LEMMA 3.7.** *Let  $B_m$ ,  $m \geq 0$ , be as defined in (3.3) and (3.4). Then, for each  $m \geq 0$ , exactly one of the following must apply:*

$$f(\overline{B}_m) = \overline{B}_{m+1}, \tag{3.5}$$

or

$$f(\overline{B}_m) = \overline{H}_{m+1}. \tag{3.6}$$

Furthermore, (3.6) holds for  $m = 0$  and for infinitely many  $m$ .

*Proof.* Note first that, since  $f$  maps compact sets to compact sets and is an open mapping, we have

$$\partial f(\overline{B}_m) \subset f(\partial \overline{B}_m). \tag{3.7}$$

Now if  $m \geq 1$ , then clearly  $\overline{H}_m = H_{m-1} \cup \overline{B}_m$  and so, by Lemma 3.3(c),

$$\overline{H}_{m+1} = H_m \cup f(\overline{B}_m).$$

We thus have

$$\overline{B}_{m+1} \subset f(\overline{B}_m) \subset \overline{H}_{m+1}. \quad (3.8)$$

But  $f(\partial \overline{B}_m) = L_{m+1} \cup L_m$ , so (3.7) implies that if  $f$  maps any point of  $\overline{B}_m$  into  $H_m$ , then  $f(\overline{B}_m)$  must contain the whole of  $H_m$ . Taken together with (3.8), this shows that (3.5) and (3.6) are the only possibilities for  $m \geq 1$ . Note also that  $f(\overline{B}_0) = \overline{H}_1$ , so (3.6) applies when  $m = 0$ .

Now suppose that (3.6) held for only finitely many  $m$ . Then, for sufficiently large  $k$ , we would have

$$f(\mathbb{C} \setminus H_{k-1}) = \bigcup_{m \geq k} f(\overline{B}_m) \subset \mathbb{C} \setminus H_{k-1},$$

so  $\mathbb{C} \setminus H_{k-1}$  would lie in the Fatou set, which is impossible.  $\square$

It follows from Lemma 3.7 that there is a strictly increasing sequence of integers  $m(j), j \geq 0$ , with  $m(0) = 0$ , such that (3.6) holds if and only if  $m = m(j)$ , for  $j \geq 0$ .

We need the following lemma [72, Lemma 1]. Here we use only the first part, but we will need the full result later so we quote it here.

**LEMMA 3.8.** *Let  $E_n, n \geq 0$ , be a sequence of compact sets in  $\mathbb{C}$ , and let  $f : \mathbb{C} \rightarrow \widehat{\mathbb{C}}$  be a continuous function such that*

$$f(E_n) \supset E_{n+1}, \text{ for } n \geq 0.$$

*Then there exists  $\zeta$  such that  $f^n(\zeta) \in E_n$ , for  $n \geq 0$ .*

*If  $f$  is also meromorphic and  $E_n \cap J(f) \neq \emptyset$  for  $n \geq 0$ , then there exists  $\zeta \in J(f)$  such that  $f^n(\zeta) \in E_n$ , for  $n \geq 0$ .*

We now describe a rule for constructing sequences of non-negative integers,  $\underline{s} = s_0 s_1 s_2 \dots$ , such that

- the itinerary of any point  $z \in \mathbb{C}$  satisfies the rule, and

*Constructing  
integer sequences  
corresponding to  
itineraries*

- with limited exceptions, any integer sequence constructed according to the rule corresponds to the itinerary of some point  $z \in \mathbb{C}$ .

Our rule is that, for each  $n \geq 0$ , we derive  $s_{n+1}$  from  $s_n$  as follows:

- (1) if  $s_n = m(j)$  for some  $j \geq 0$ , then

$$s_{n+1} \in \{0, 1, 2, \dots, m(j), m(j) + 1\};$$

- (2) otherwise,  $s_{n+1} = s_n + 1$ .

The itinerary of any point  $z \in \mathbb{C}$  satisfies this rule by Lemma 3.7, since:

$$f(L_m) = L_{m+1}, \quad m \geq 0. \quad (3.9)$$

On the other hand, if  $\underline{s}$  is an integer sequence constructed according to this rule, and we put  $E_n = \overline{B}_{s_n}$  for  $n \geq 0$ , then it follows from Lemma 3.7 that the sequence of compact sets  $(E_n)_{n \geq 0}$  and the function  $f$  satisfy the conditions of Lemma 3.8. Hence there exists a point  $z \in E_0 = \overline{B}_{s_0}$  such that  $f^n(z) \in E_n = \overline{B}_{s_n}$ , for  $n \geq 0$ .

However, the itineraries of points are defined relative to the sets  $B_m$ ,  $m \geq 0$ , which partition the plane, rather than relative to the compact sets  $\overline{B}_m = B_m \cup L_m$  we have used in the construction of points corresponding to integer sequences. However, if any iterate of a point lies in  $L_m$  for some  $m \geq 0$ , then all subsequent iterates also lie in a fundamental loop by (3.9). Thus the only situation in which an integer sequence constructed according to our rule may not coincide with the itinerary of a point derived from the sequence by using Lemma 3.8 is where the orbit of the point ends on the fundamental loops  $(L_m)_{m \geq 0}$ .

In particular, since  $L_m \subset A(f)$ ,  $m \geq 0$ , if an integer sequence  $\underline{s}$  gives rise to a point  $z$  in  $A(f)^c$ , then the itinerary of  $z$  is  $\underline{s}$ . Furthermore, any two points in  $A(f)^c$  with different itineraries must necessarily lie in different components of  $A(f)^c$ , so all points in the same component of  $A(f)^c$  as  $z$  have itinerary  $\underline{s}$ .



We are now in a position to prove Theorem 1.3 which states that, if  $A_R(f)$  is a spider's web, then  $A(f)^c$  has uncountably many components

- (a) whose orbits are bounded,
- (b) whose orbits are unbounded but contain a bounded suborbit,  
and
- (c) whose orbits escape to infinity.

*Proof of Theorem 1.3.* We examine each of the orbit types (a) - (c) in turn, showing how to construct an itinerary for a point in a component of  $A(f)^c$  with that type of orbit, and proving that there must be uncountably many such components. Note that many alternative constructions are possible for each orbit type.

For type (a), components with bounded orbit, we can construct an itinerary in the following way:

*Components of  
 $A(f)^c$  with  
bounded orbit*

- choose  $j_0 \geq 2$ , and put  $s_0 = m(j_0)$ ;
- for  $n \geq 0$ :
  - (i) if  $s_n = m(j_0)$ , put  $s_{n+1} = m(j_0) - 1$ ;
  - (ii) otherwise, put  $s_{n+1} = s_n + 1$ .

Evidently, by Lemma 3.8 and the ensuing discussion, we thereby obtain a point  $\alpha \in B_{m(j_0)} \cap A(f)^c$  whose orbit is bounded.

To prove that there are uncountably many such points, we use an idea from a proof by Milnor [56, Corollary 4.15, p.49]. Given any finite partial itinerary  $s_0 s_1 \dots s_k$  corresponding to the first  $k$  iterations of the point  $\alpha$ , then for the next value of  $n > k$  for which  $s_n = m(j_0)$ , instead of assigning  $s_{n+1}$  the value  $m(j_0) - 1$  under (i) above, we could instead put  $s_{n+1} = m(j_0) - 2$ . The remaining  $s_n$  are then chosen as above. By Lemma 3.8, this sequence gives rise to another point  $\alpha' \in B_{m(j_0)} \cap A(f)^c$  with the same finite partial itinerary  $s_0 s_1 \dots s_k$  as  $\alpha$ , but with an ultimately different bounded orbit.

Thus the finite partial itinerary  $s_0 s_1 \dots s_k$  can be extended in two different ways to yield two further finite partial itineraries, each of which may again be extended in the same way. By continuing this process, it follows that  $s_0 s_1 \dots s_k$  can be extended in uncountably many

ways, and Lemma 3.8 shows that each resulting infinite itinerary corresponds to a distinct point in  $B_{m(j_0)} \cap A(f)^c$ . Since any two points in  $A(f)^c$  with different itineraries must lie in different components of  $A(f)^c$ , it follows that there are uncountably many components of  $A(f)^c$  with bounded orbits.

*Components of  
 $A(f)^c$  with  
 unbounded orbit but  
 a bounded suborbit*

To construct an itinerary of type (b), i.e. for a component of  $A(f)^c$  whose orbit is unbounded but contains a bounded suborbit, we can proceed as follows:

- put  $s_0 = 0$ ;
- for  $n \geq 0$ :
  - (i) if there exists  $j \geq 2$  such that  $s_n = m(j)$  and  $s_i \neq m(j)$  for  $i = 0, 1, 2, \dots, n-1$ , put  $s_{n+1} = 0$ ;
  - (ii) otherwise, put  $s_{n+1} = s_n + 1$ .

By Lemma 3.8, we thereby obtain a point  $b \in B_0 \cap A(f)^c$  whose orbit is unbounded, but which visits  $B_0$  infinitely often. Evidently, at any stage when the orbit returns to  $B_0$ , we could equally well have returned it to  $B_1$ , and it therefore follows by the same argument as for type (a) that there are uncountably many components of  $A(f)^c$  with orbits of type (b).

*Components of  
 $A(f)^c$  whose orbits  
 escape to infinity*

Finally, consider type (c), i.e. components of  $A(f)^c$  whose orbits escape to infinity.

For each  $i \in \mathbb{N}$ , let  $j_i$  be the largest value of  $j$  such that

$$\tilde{B}_{m(j)} \subset \{z : |z| < M^i(R)\},$$

or, if no such values of  $j$  exist, let  $j_i = 0$ . Let  $I$  be the smallest value of  $i$  for which  $j_i \neq 0$ .

To construct an itinerary of type (c), our procedure is:

- put  $s_0 = m(j_I)$ ;
- for  $n \geq 0$ :
  - (i) if  $s_n = m(j_i)$  for some  $i \geq I$ , and if  $n \leq 2i - I$ , put  $s_{n+1} = m(j_i)$ ;
  - (ii) otherwise, put  $s_{n+1} = s_n + 1$ .

The purpose of this construction is to keep the orbit of the constructed point within the closure of  $\tilde{B}_{m(j_i)}$ ,  $i \geq I$ , until at least  $2i - I$  iterations have taken place. To see that a point  $z$  with such an itinerary lies in  $A(f)^c$ , note that for all  $i \geq I$ ,

$$|f^{2i-I}(z)| < M^i(R). \quad (3.10)$$

It follows that there is no value of  $\ell \in \mathbb{N}$  such that

$$|f^{i+\ell}(z)| \geq M^i(R), \text{ for all } i \in \mathbb{N},$$

since putting  $i = I + \ell$  contradicts (3.10). Thus, from the definition,  $z \notin A(f)$ .

Hence, by Lemma 3.8, we obtain a point  $c \in B_{m(j_I)} \cap A(f)^c$  which escapes to infinity. If we are given any finite partial itinerary  $s_0 s_1 \dots s_k$  corresponding to the first  $k$  iterations of  $c$ , then for the next value of  $n > k$  such that, for some  $i \geq I$ ,

$$s_n = m(j_i) \text{ and } n = 2i - I + 1,$$

instead of applying (ii) above, we could equally put  $s_{n+1} = m(j_i)$ . The finite partial itinerary  $s_0 s_1 \dots s_k$  can therefore be extended in two different ways to yield two further finite partial itineraries, corresponding to two different points in  $B_{m(j_I)} \cap A(f)^c$  with the same initial iteration sequence, but with ultimately different orbits escaping to infinity. Thus, using the same argument as previously, there are uncountably many components of  $A(f)^c$  with orbits of type (c). This completes the proof.  $\square$

REMARK. The method of proof of Theorem 1.3 can also be applied to show the existence of components of  $A(f)^c$  with other types of orbits. For example, using [72, Theorem 1], we can adapt the proof for orbits of type (c) to show that, if  $f$  is a transcendental entire function such that  $A_R(f)$  is a spider's web, then there are uncountably many components  $K$  of  $A(f)^c$  whose orbits escape to infinity arbitrarily slowly, in the sense that if  $(a_n)$  is any positive sequence such that  $a_n \rightarrow \infty$  as  $n \rightarrow \infty$ , then  $|f^n(z)| \leq a_n$  for sufficiently large  $n$  and for all  $z \in K$ .

*Components of  
 $A(f)^c$  with other  
types of orbit*

3.5 COMPONENTS OF  $A(f)^c$  WITH BOUNDED ORBITS

In this section, we again assume that  $f$  is a transcendental entire function such that  $A_R(f)$  is a spider's web, and we examine further the components of  $A(f)^c$  with bounded orbits. We show that, in this case, we can say much more about the nature of such components than is given by Theorem 1.2. We do this by following a method used by Kisaka [46] and, in a different context, by Eremenko and Lyubich [36] and by Zheng [91]. The method makes use of polynomial-like mappings - see Section 2.4 for the definition of a polynomial-like mapping and its filled Julia set, and a brief introduction to their properties.

*Kisaka's results on  
components of  $J(f)$   
when  $f$  has a  
multiply connected  
Fatou component*

Building on results in [45], Kisaka proved in [46, Theorem A] that, if  $f$  is a transcendental entire function with a multiply connected Fatou component, and  $C$  is a component of  $J(f)$  with bounded orbit, then there is a polynomial  $g$  such that  $C$  is quasiconformally homeomorphic to a component of the Julia set of  $g$ . Moreover, he proved that

- (i) if the complement of  $C$  is connected, then  $C$  is a buried component of  $J(f)$ , and
- (ii) if  $C$  is a wandering component of  $J(f)$ , then  $C$  is a buried singleton component of  $J(f)$ .

Eremenko and Lyubich [36] and Zheng [91] used a similar technique to obtain results about Fatou components with unbounded orbits for certain transcendental entire functions (see our remark following the proof of Theorem 1.6 below for further details).

*Analogous results  
for components of  
 $A(f)^c$  when  $A_R(f)$   
is a spider's web*

We now prove results analogous to those of Kisaka, but expressed in terms of components of  $A(f)^c$  rather than of  $J(f)$ , and with  $f$  belonging to the wider class of transcendental entire functions for which  $A_R(f)$  is a spider's web.

Note that Theorem 1.2(b) and (c) already gives us an analogue of (i) in Kisaka's result. Indeed, it does more, for there we do not assume that the component of  $A(f)^c$  has bounded orbit.

We now prove Theorem 1.5, which establishes the existence of a quasiconformal conjugacy with a polynomial for components of  $A(f)^c$  with bounded orbit when  $A_R(f)$  is a spider's web.

*Proof of Theorem 1.5.* Let  $K$  be a component of  $A(f)^c$  with bounded orbit, and let the sequences of fundamental holes and loops for  $A_R(f)$  be  $(H_n)_{n \geq 0}$  and  $(L_n)_{n \geq 0}$  respectively.

*Proof of the  
quasiconformal  
conjugacy with a  
polynomial*

Since  $f$  is transcendental and the orbit of  $K$  is bounded, it follows from Lemma 3.3 parts (a) and (c) that we may choose  $m \in \mathbb{N}$  so large that the orbit of  $K$  lies in  $H_m$ , and such that  $f$  is a proper map of  $H_m$  onto  $H_{m+1}$  of degree at least 2. It then follows from Lemma 3.3(d) that there exists  $N \in \mathbb{N}$  such that

- $f^N$  is a proper map of  $H_m$  onto  $H_{m+N}$  and of  $H_{m+N}$  onto  $H_{m+2N}$  of degree at least 2, and
- we have  $L_{m+N} \cap L_m = \emptyset$  and  $L_{m+2N} \cap L_{m+N} = \emptyset$ .

Now let  $\gamma$  be a smooth Jordan curve in  $H_{m+2N}$  that surrounds  $\overline{H_{m+N}}$  and does not meet any of the critical values of  $f^N$ , and let  $V$  be the bounded component of  $\gamma^c$ , so that

$$H_m \subset \overline{H_{m+N}} \subset V \subset H_{m+2N}.$$

Define  $U$  to be the component of  $f^{-N}(V)$  that contains  $H_m$  (and hence the orbit of  $K$ ). Then  $U$  lies in the component of  $f^{-N}(H_{m+2N})$  that contains  $H_m$ , i.e.  $U \subset H_{m+N}$ , and so we have  $\overline{U} \subset V$ . Furthermore,  $U$  is simply connected, and  $f^N : U \rightarrow V$  is a proper map of degree at least 2. Since  $V$  is bounded by a smooth Jordan curve that does not meet any of the critical values of  $f^N$ , it follows that  $U$  is also bounded by a smooth Jordan curve. We have therefore established that the conditions of Definition 2.11 are satisfied, so the triple  $(f^N; U, V)$  is a polynomial-like mapping of degree at least 2.

Now the set  $\overline{U}$  consists of a collection of components (or parts of components) of  $A(f)^c$ , together with a bounded subset of  $A(f)$ . Clearly points in  $A(f)$  cannot lie in the filled Julia set  $K(f^N; U, V)$  of the polynomial-like mapping  $(f^N; U, V)$ , but points in  $A(f)^c$  may do

so. In particular, since the orbit of the component  $K$  under iteration by  $f$  lies in  $U$ , it must also lie in  $K(f^N; U, V)$ .

Indeed, since  $f$  maps every component of  $A(f)^c$  onto another such component (Theorem 3.6), and points in  $A(f)$  cannot lie in  $K(f^N; U, V)$ , it follows that every component of  $A(f)^c$  in the orbit of  $K$  must be a distinct component of  $K(f^N; U, V)$ . Now, by Douady and Hubbard's Straightening Theorem (Theorem 2.12), there is a polynomial  $g$  of degree at least 2 such that  $K(f^N; U, V)$  is quasiconformally homeomorphic to the filled Julia set of  $g$ , and thus it follows that each component of  $A(f)^c$  in the orbit of  $K$  is quasiconformally homeomorphic to a component of the filled Julia set of  $g$ .  $\square$

The existence of the quasiconformal mapping in Theorem 1.5 enables us to use the following recent results from polynomial dynamics to draw some further conclusions.

*Recent results from  
polynomial  
dynamics*

**THEOREM 3.9** (Kozlovski and van Strien [48], Qiu and Yin [65]). *For a non-linear polynomial  $g$ , a component of the filled Julia set  $K(g)$  is a singleton if and only if its orbit includes no periodic component of  $K(g)$  containing a critical point.*

**THEOREM 3.10** (Roesch and Yin [76, 77]). *If  $g$  is a non-linear polynomial, then any bounded component of  $F(g)$  which is not a Siegel disc is a Jordan domain.*

*Restatement of  
Theorem 1.6*

As before, let  $f$  be a transcendental entire function, let  $R > 0$  be such that  $M(r, f) > r$  for  $r \geq R$ , and let  $A_R(f)$  be a spider's web. We prove Theorem 1.6, which states the following.

- (a) If  $K$  is a component of  $A(f)^c$  with bounded orbit, then
- (i) the component  $K$  is a singleton if and only if the orbit of  $K$  includes no periodic component of  $A(f)^c$  containing a critical point. In particular, if  $K$  is a wandering component of  $A(f)^c$ , then  $K$  is a singleton.
  - (ii) the interior of  $K$  is either empty or consists of non-wandering Fatou components. If these Fatou components are not Siegel discs, then they are Jordan domains.

- (b) All except at most countably many of the components of  $A(f)^c$  with bounded orbits are singletons.

Note that part (a)(i) of Theorem 1.6 is an analogue of (ii) in Kisaka's result.

*Proof of Theorem 1.6.* Let  $K$  be a component of  $A(f)^c$  with bounded orbit. Then, by Theorem 1.5, any component of  $A(f)^c$  in the orbit of  $K$  is quasiconformally homeomorphic to a component of the filled Julia set of some non-linear polynomial  $g$ . Since periodic orbits and critical points are preserved by the homeomorphism, it follows from Theorem 3.9 that  $K$  is a singleton if and only if its orbit includes no periodic component of  $A(f)^c$  containing a critical point. Wandering components of  $A(f)^c$  clearly have no periodic components in their orbit, so if  $K$  is a wandering component, then it must be a singleton. This proves part (a)(i).

To prove part (a)(ii), let  $K$  again denote a component of  $A(f)^c$  with bounded orbit. If the interior of  $K$  is non-empty, then by Theorem 1.2 the interior must consist of one or more components of  $F(f)$ . Now the quasiconformal homeomorphism obtained in the proof of Theorem 1.5 maps the interior of a component of  $A(f)^c$  onto the interior of a component of the filled Julia set of a non-linear polynomial  $g$ , which consists of Fatou components of  $g$  that must be non-wandering by Sullivan's theorem [85]. Part (a)(ii) now follows immediately from Theorem 3.10, since Siegel discs and Jordan curves are clearly preserved by the homeomorphism.

Part (b) follows from part (a)(i) because  $f$  has only countably many critical points.  $\square$

REMARKS. 1. In [91, Theorem 3], Zheng used a method similar to that adopted in the proof of Theorem 1.5 to show that, if  $f$  is a transcendental entire function with a multiply connected Fatou component, and if  $U$  is any wandering Fatou component, then there exists a subsequence  $f^{n_k}$  of  $(f^n)_{n \in \mathbb{N}}$  such that  $f^{n_k}|_U \rightarrow \infty$  as  $k \rightarrow \infty$ . Thus, the orbit of every wandering Fatou component is unbounded.

*Zheng's results on wandering Fatou components*

Note that it follows from our Theorem 1.6(a)(ii) that the orbit of every wandering Fatou component is unbounded whenever  $A_R(f)$  is a spider's web.

In [91, Theorem 4], Zheng used the same method as in his Theorem 3 to show that every wandering Fatou component has an unbounded orbit for transcendental entire functions such that

$$m(r, f) > r, \text{ for an unbounded sequence of } r. \quad (3.11)$$

A discussion of the same idea also appears in Eremenko and Lyubich [36]. The proof of this result can readily be adapted to show that the orbit of every wandering Fatou component is unbounded if  $f$  is *strongly polynomial-like* (see Definition 1.9 and Section 4.3). This result would then cover those transcendental entire functions for which (3.11) holds, as well as those for which  $A_R(f)$  is a spider's web.

2. It follows from Theorem 1.6(a)(ii) that, if  $f$  is a transcendental entire function such that  $A_R(f)$  is a spider's web, then the boundary of each Fatou component in an attracting or parabolic basin is a Jordan curve. As an example of this, the function

$$f(z) = \frac{1}{2}(\cos z^{1/4} + \cosh z^{1/4})$$

has an  $A_R(f)$  spider's web and also a real attracting fixed point whose immediate basin of attraction must be bounded by a Jordan curve; see [73, Figure 1] for an illustration of  $A_R(f)$  for the above function which shows this basin.

### 3.6 PERIODIC COMPONENTS OF $A(f)^c$

*Singleton periodic components of  $A(f)^c$  are dense in the Julia set*

In this section we prove Theorem 1.7 which states that, if  $A_R(f)$  is a spider's web, then  $A(f)^c$  has singleton periodic components and these components are dense in  $J(f)$ .

Domínguez [28] has shown that, if  $f$  is a transcendental entire function with a multiply connected Fatou component, then  $J(f)$  has buried



singleton components, and such components are dense in  $J(f)$  (see also [29]). Our proof of Theorem 1.7 is based on earlier results from this chapter and on the method used by Bergweiler [16] in his alternative proof of Domínguez' result. The method relies on a corollary of the Ahlfors five islands theorem - see Section 2.5 for a brief introduction to this theorem and a statement of the corollary (Proposition 2.14).

*Proof of Theorem 1.7.* It follows from Theorems 1.3(a) and 1.6(b) that, if  $A_R(f)$  is a spider's web, then there are uncountably many singleton components of  $A(f)^c$ , and by Theorem 1.2 these lie in  $J(f)$ .

Now  $J(f)$  is the closure of the backwards orbit  $O^-(z)$  of any non-exceptional point  $z \in J(f)$ . Since  $f$  is an open mapping and  $A(f)$  is completely invariant, the preimages of singleton components of  $A(f)^c$  are themselves singleton components of  $A(f)^c$ , and it therefore follows that singleton components of  $A(f)^c$  are dense in  $J(f)$ .

We now claim that singleton *periodic* components of  $A(f)^c$  are dense in  $J(f)$ .

To prove this, let  $W$  be any neighbourhood of a point  $w \in J(f)$ . Then since  $J(f)$  is perfect,  $W$  contains infinitely many points in  $J(f)$ . Thus there exist  $w_j \in J(f)$ ,  $j \in \{1, 2, 3\}$  and  $\varepsilon > 0$  such that the Jordan domains  $D_j = B(w_j, \varepsilon)$  have pairwise disjoint closures and lie in  $W$ .

Now since singleton components of  $A(f)^c$  are dense in  $J(f)$ , for  $j \in \{1, 2, 3\}$  there exist singleton components  $\{z_j\}$  of  $A(f)^c$  such that  $z_j \in D_j$ . Moreover, by Theorem 1.2(b), there are closed subsets  $X_j$  of  $A(f) \cap J(f)$  lying in  $D_j$  and surrounding  $z_j$ . Let  $V_j$  be the bounded complementary component of  $X_j$  containing  $z_j$ . Then  $\partial V_j \subset A(f)$  and, since  $z_j \in J(f)$ , it follows that  $V_1, V_2, V_3$  are domains satisfying  $V_j \cap J(f) \neq \emptyset$  for  $j \in \{1, 2, 3\}$ . Thus we may apply Proposition 2.14, obtaining  $\mu \in \{1, 2, 3\}$ ,  $n \in \mathbb{N}$  and a domain  $U \subset V_\mu$  such that  $f^n : U \rightarrow D_\mu$  is conformal.

Now let  $\phi$  be the branch of the inverse function  $f^{-n}$  which maps  $D_\mu$  onto  $U$ . Then  $\phi$  must have a fixed point  $z_0 \in U \subset V_\mu$ . Furthermore, by the Schwarz lemma, this fixed point must be attracting, and

because  $\phi(D_\mu) = U$  where  $\bar{U}$  is a compact subset of  $D_\mu$ , we have that  $\phi^k(z) \rightarrow z_0$  as  $k \rightarrow \infty$ , uniformly for  $z \in D_\mu$ .

Since  $z_0$  is an attracting fixed point of  $\phi$ , it is a repelling fixed point of  $f^n$  and hence a repelling periodic point of  $f$ . Thus  $z_0$  lies in  $J(f) \cap A(f)^c$ .

Now  $z_0 = \phi^k(z_0) \in \phi^k(V_\mu)$  for all  $k \in \mathbb{N}$ , and  $\text{diam } \phi^k(\bar{V}_\mu) \rightarrow 0$  as  $k \rightarrow \infty$ . It follows that

$$\bigcap_{k \in \mathbb{N}} \phi^k(\bar{V}_\mu) = \{z_0\}. \quad (3.12)$$

Since  $\partial V_\mu$  lies in  $A(f)$ , which is completely invariant, and  $\phi$  is conformal, we have  $\partial \phi^k(V_\mu) = \phi^k(\partial V_\mu) \subset A(f)$  for all  $k \in \mathbb{N}$ . But  $\phi^k(\partial V_\mu)$  surrounds  $z_0$  for all  $k \in \mathbb{N}$ , so  $\{z_0\}$  must be a singleton component of  $A(f)^c$  by (3.12).

We have therefore shown that, in any neighbourhood of an arbitrary point of  $J(f)$ , there is a singleton component of  $A(f)^c$  that is also a repelling periodic point of  $f$ . This proves the claim.

To complete the proof of the theorem, note finally that if  $f$  has a multiply connected Fatou component, then it follows from Theorem 1.2(c) that the singleton periodic components of  $A(f)^c$  constructed above are buried components of  $J(f)$ .  $\square$

## CONNECTEDNESS PROPERTIES OF THE SET OF POINTS WHOSE ORBITS ARE BOUNDED

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In this chapter, we prove our results on the connectedness properties of the set

$$K(f) = \{z \in \mathbb{C} : (f^n(z))_{n \in \mathbb{N}} \text{ is bounded}\},$$

that is, the set of points whose orbits are bounded under iteration, where  $f$  is a transcendental entire function. In particular, we prove new results on the number of components of  $K(f)$ , and we exhibit a class of transcendental entire functions (which we call strongly polynomial-like functions) for which  $K(f)$  is totally disconnected if and only if each component of  $K(f)$  containing a critical point is aperiodic.

Recall from Definition 1.9 that a transcendental entire function  $f$  is *strongly polynomial-like* if there exist sequences  $(V_n), (W_n)$  of bounded, simply connected domains with smooth boundaries such that

$$V_n \subset V_{n+1} \text{ and } W_n \subset W_{n+1} \text{ for } n \in \mathbb{N}, \quad (4.1)$$

$$\bigcup_{n \in \mathbb{N}} V_n = \bigcup_{n \in \mathbb{N}} W_n = \mathbb{C}, \quad (4.2)$$

and each of the triples  $(f; V_n, W_n)$  is a polynomial-like mapping (for the definition of a polynomial-like mapping, see Section 2.4).

The organisation of this chapter is as follows. In Section 4.1, we prove Theorem 1.11 and Corollary 1.12, on the properties of components of  $K(f)$  for strongly polynomial-like functions. Section 4.2 contains the proofs of our results on the number of components of  $K(f)$  (Theorems 1.8 and 1.10). In Section 4.3 we prove Theorem 1.13, which gives an alternative characterisation of strongly polynomial-like functions, and Theorem 1.14, which exhibits several types of function with this property. In Section 4.4, we give examples of transcendental en-

*Organisation of  
the chapter*

tire functions for which  $K(f)$  is totally disconnected. Finally, in Section 4.5, we use quasiconformal surgery to construct a transcendental entire function for which  $K(f)$  has a component with empty interior which is not a singleton.

*Relationship with  
results where  $A_R(f)$   
is a spider's web*

REMARK. In the following notes, we clarify the relationship between the results proved in this chapter for strongly polynomial-like functions, and earlier results for transcendental entire functions with the property that  $A_R(f)$  is a spider's web.

- It follows from Lemma 3.3 and Theorem 1.13 that, if  $A_R(f)$  is a spider's web, then  $f$  is strongly polynomial-like. However, the converse is not true - see [75, Theorem 1.2].
- Theorem 1.14 is similar to [73, Theorem 1.9], which gave various classes of functions for which  $A_R(f)$  is a spider's web. However, in Theorem 1.14 we do not need the additional regular growth condition that was required for several of the function classes in [73, Theorem 1.9].
- We claim that, if  $A_R(f)$  is a spider's web, then  $K$  is a component of  $K(f)$  if and only if  $K$  is a component of  $A(f)^c$  with bounded orbit. Thus Theorem 1.11 is a generalisation to strongly polynomial-like functions of results previously proved for functions with an  $A_R(f)$  spider's web in Theorem 1.6.

To show that the claim is true, assume that  $A_R(f)$  is a spider's web and let  $K$  be a component of  $K(f)$ . Then the orbit of  $K$  must lie in some bounded, simply connected domain  $G$  whose boundary is in  $A(f)$ . Clearly  $K \subset A$  for some component  $A$  of  $A(f)^c$ , and if  $A \neq K$  then  $A \setminus K$  must contain a point  $z$  whose orbit is unbounded. However, we would then have that, for some  $n \in \mathbb{N}$ ,  $f^n(K) \subset G$  but  $f^n(z) \in \mathbb{C} \setminus G$ , which is a contradiction because  $f^n(A)$  cannot meet  $\partial G$ . Since it is easy to see that every component of  $A(f)^c$  with bounded orbit is a component of  $K(f)$ , this establishes the claim.

## 4.1 PROOFS OF THEOREM 1.11 AND COROLLARY 1.12

We now prove our results on the properties of components of  $K(f)$  for strongly polynomial-like functions (Theorem 1.11 and Corollary 1.12).

Our proof makes use of the following topological result.

**LEMMA 4.1.** *A countable union of compact, totally disconnected subsets of  $\mathbb{C}$  is totally disconnected.*

*Proof.* This is an immediate consequence of the following results, which may be found in Hurewicz and Wallman [42, Chapter II]:

- a compact, separable metric space is totally disconnected if and only if it is 0-dimensional;
- if a separable metric space is the countable union of 0-dimensional closed subsets of itself, then it is 0-dimensional;
- every 0-dimensional, separable metric space is totally disconnected.

Here, a non-empty space is *0-dimensional* if each of its points has arbitrarily small neighbourhoods with empty boundaries.  $\square$

We also need the recent results from polynomial dynamics quoted in Chapter 3, namely Theorem 3.9, the proof of the Branner-Hubbard conjecture (a component of the filled Julia set of a non-linear polynomial is a singleton if and only if its orbit includes no periodic component containing a critical point) and Theorem 3.10 (any bounded Fatou component of a non-linear polynomial which is not a Siegel disc is a Jordan domain).

We are now in a position to prove Theorem 1.11 which says that, if  $f$  is a strongly polynomial-like transcendental entire function and  $K$  is a component of  $K(f)$ , then the following properties hold.

*Restatement of  
Theorem 1.11*

- (a) The component  $K$  is a singleton if and only if the orbit of  $K$  includes no periodic component of  $K(f)$  containing a critical point. In particular, if  $K$  is a wandering component of  $K(f)$ , then  $K$  is a singleton.

- (b) The interior of  $K$  is either empty or consists of bounded, non-wandering Fatou components. If these Fatou components are not Siegel discs, then they are Jordan domains.

*Proof of Theorem 1.11.* Since  $f$  is strongly polynomial-like, it follows that there exist sequences  $(V_n), (W_n)$  of bounded, simply connected domains with smooth boundaries satisfying (4.1) and (4.2), and such that each of the triples  $(f; V_n, W_n)$  is a polynomial-like mapping.

Let  $K(f; V_n, W_n)$  denote the filled Julia set of the polynomial-like mapping  $(f; V_n, W_n)$ . Then clearly we have

$$K(f; V_n, W_n) \subset K(f; V_{n+1}, W_{n+1}), \quad \text{for } n \in \mathbb{N},$$

and

$$K(f) = \bigcup_{n \in \mathbb{N}} K(f; V_n, W_n). \quad (4.3)$$

*A necessary and  
sufficient condition  
for a component of  
 $K(f)$  to be a  
singleton*

To prove part (a), first let  $K$  be a component of  $K(f)$  whose orbit includes no periodic component of  $K(f)$  containing a critical point. We show that  $K$  must be a singleton.

For each  $n \in \mathbb{N}$ , define

$$K_n = K \cap K(f; V_n, W_n).$$

Then  $K = \bigcup_{n \in \mathbb{N}} K_n$ , and since any component of  $K(f; V_n, W_n)$  must lie in a single component of  $K(f)$  it follows that, where  $K_n \neq \emptyset$ , each component of  $K_n$  must be a component of  $K(f; V_n, W_n)$ . In particular, each component of  $K_n$  must be compact.

Moreover, no component of  $K_n$  can have an orbit which includes a periodic component of  $K(f; V_n, W_n)$  containing a critical point. For any such periodic component of  $K(f; V_n, W_n)$  would lie in a periodic component of  $K(f)$ , and since  $K_n \subset K$ , the orbit of  $K$  would then include a periodic component of  $K(f)$  containing a critical point, contrary to our assumption.

Now it follows from Douady and Hubbard's Straightening Theorem (Theorem 2.12) that, for each  $n \in \mathbb{N}$ , there exists a quasiconfor-

mal mapping  $\phi_n : \mathbb{C} \rightarrow \mathbb{C}$  and a polynomial  $g_n$  of the same degree as  $(f; V_n, W_n)$  such that  $\phi_n \circ f = g_n \circ \phi_n$  on  $\bar{V}_n$ , and

$$\phi_n(K(f; V_n, W_n)) = K(g_n), \quad (4.4)$$

where  $K(g_n)$  is the filled Julia set of the polynomial  $g_n$ .

Thus it follows from (4.4) and Theorem 3.9, and the fact that critical points are preserved by the quasiconformal mapping, that every component of  $K_n$  is a singleton, i.e.  $K_n$  is totally disconnected, for each  $n \in \mathbb{N}$ . Lemma 4.1 now gives that  $K$  is totally disconnected, and since  $K$  is connected it must be a singleton.

For the converse, suppose now that a component  $K$  of  $K(f)$  is a singleton. Then it follows from (4.3) that there exists  $N \in \mathbb{N}$  such that  $K$  is a singleton component of  $K(f; V_n, W_n)$  for all  $n \geq N$ . Thus, by (4.4) and Theorem 3.9, for each  $n \geq N$  the orbit of  $K$  can include no periodic component of  $K(f; V_n, W_n)$  containing a critical point. The desired converse now follows from (4.3).

Finally, since by definition the orbit of a wandering component of  $K(f)$  contains no periodic component, it follows that every wandering component of  $K(f)$  is a singleton. This completes the proof of part (a).

To prove part (b) note first that, for any transcendental entire function  $f$ , since  $J(f) = \partial K(f)$  it is immediate that for any component  $K$  of  $K(f)$  we have  $\partial K \subset J(f)$  and  $\text{int}(K) \subset F(f)$ .

Now let  $f$  be strongly polynomial-like, and let  $K$  be a component of  $K(f)$  with non-empty interior. As in the proof of part (a), we write  $K = \bigcup_{n \in \mathbb{N}} K_n$  where

$$K_n = K \cap K(f; V_n, W_n),$$

so  $K_n$  has non-empty interior for sufficiently large  $n$ . Then, since every component of  $K_n$  is a component of  $K(f; V_n, W_n)$ , it follows from (4.4) that the interior of a component of  $K_n$  is quasiconformally homeomorphic to the interior of a component of the filled Julia set  $K(g_n)$  of the polynomial  $g_n$ , which consists of bounded Fatou components that are non-wandering by Sullivan's theorem [85]. Evidently, therefore, if a Fatou component  $U$  of  $f$  meets  $K_n$ , we have  $\bar{U} \subset K_n$ , and

*Properties of the interior and the boundary of components of  $K(f)$*

it follows that all Fatou components in  $K(f)$  are bounded and non-wandering. Since Siegel discs and Jordan curves are preserved by the quasiconformal mapping, the remainder of part (b) now follows from Theorem 3.10.  $\square$

We now prove Corollary 1.12 which states that, if  $f$  is a strongly polynomial-like transcendental entire function then

- (a) all except at most countably many components of  $K(f)$  are singletons, and
- (b)  $K(f)$  is totally disconnected if and only if each component of  $K(f)$  containing a critical point is aperiodic.

*Proof of Corollary 1.12.* Since  $f$  is strongly polynomial-like, it follows from Theorem 1.11(a) that a component  $K$  of  $K(f)$  is a singleton unless the orbit of  $K$  includes a periodic component of  $K(f)$  containing a critical point. Part (a) now follows because  $f$  can have at most countably many critical points.

If  $K(f)$  is totally disconnected then all of its components are singletons, so part (b) follows immediately from Theorem 1.11(a).  $\square$

*Relationship to some  
results of Zheng*

REMARK. Zheng [90, Theorem 2], [91, Theorem 4] has shown that, if  $f$  is a transcendental entire function for which there exists an unbounded sequence  $(r_n)$  of positive real numbers such that

$$m(r_n, f) > r_n, \quad \text{for } n \in \mathbb{N},$$

and if  $U$  is a component of  $F(f)$ , then

- (i) if  $U$  contains a point  $z_0$  such that  $\{f^n(z_0) : n \in \mathbb{N}\}$  is bounded, then  $U$  is bounded, and
- (ii) if  $U$  is wandering, then there exists a subsequence of  $f^n$  on  $U$  tending to  $\infty$ .

It follows that, for such functions, the interior of  $K(f)$  consists of bounded, non-wandering Fatou components. As these functions are strongly polynomial-like by Theorem 1.14, the first part of Theorem 1.11(b) is a generalisation of Zheng's results.



## 4.2 PROOFS OF THEOREMS 1.8 AND 1.10

In this section we prove Theorems 1.8 and 1.10, which concern the number of components of  $K(f)$  and  $K(f) \cap J(f)$  when  $f$  is a transcendental entire function.

Theorem 1.8 states that, for a transcendental entire function  $f$ , the set  $K(f)$  is either connected or has infinitely many components. This is a consequence of the following result due to Rippon and Stallard. Recall that  $E(f)$  is the *exceptional set* of  $f$ , i.e. the set of points with a finite backwards orbit under  $f$  (which for a transcendental entire function contains at most one point).

*Components of  $K(f)$   
for a general  
transcendental  
entire function*

**THEOREM 4.2.** [74, Theorem 5.2] *Let  $f$  be a transcendental entire function. Suppose that the set  $S$  is completely invariant under  $f$ , and that  $J(f) = \overline{S \cap J(f)}$ . Then exactly one of the following holds:*

- (1)  $S$  is connected;
- (2)  $S$  has exactly two components, one of which is a singleton  $\{\alpha\}$ , where  $\alpha$  is a fixed point of  $f$  and  $\alpha \in E(f) \cap F(f)$ ;
- (3)  $S$  has infinitely many components.

*Proof of Theorem 1.8.* Since  $K(f)$  is completely invariant and dense in  $J(f)$ , it is evident that the conditions of Theorem 4.2 hold with  $S = K(f)$ . Case (2) cannot occur since if  $z \in F(f)$  has bounded orbit, then so does a neighbourhood of  $z$  in  $F(f)$ .  $\square$

Theorem 1.10 gives a new result on components of  $K(f) \cap J(f)$  for a general transcendental entire function, and also shows that we can improve on Theorem 1.8 for strongly polynomial-like functions. The statement of the theorem is as follows. Let  $f$  be a transcendental entire function. Then:

- (a) Either  $K(f) \cap J(f)$  is connected, or else every neighbourhood of a point in  $J(f)$  meets uncountably many components of  $K(f) \cap J(f)$ .
- (b) If  $f$  is strongly polynomial-like then either  $K(f)$  is connected, or else every neighbourhood of a point in  $J(f)$  meets uncountably many components of  $K(f)$ .

In our proof of this result, we will need to call upon a result of Rippon and Stallard quoted in Chapter 3 (Lemma 3.8), and the following topological lemma due to Rempe.

**LEMMA 4.3.** [68, Lemma 3.1] *Let  $C \subset \mathbb{C}$ . Then  $C$  is disconnected if and only if there is a closed connected set  $A \subset \mathbb{C}$  such that  $C \cap A = \emptyset$  and at least two different connected components of  $\mathbb{C} \setminus A$  intersect  $C$ .*

Components of  
 $K(f) \cap J(f)$  for a  
general  
transcendental  
entire function

*Proof of Theorem 1.10.* We first prove part (a). If  $K(f) \cap J(f)$  is disconnected, then it follows from Lemma 4.3 that there exists a continuum  $\Gamma \subset (K(f) \cap J(f))^c$  with two complementary components,  $G_1$  and  $G_2$  say, each of which contains points in  $K(f) \cap J(f)$  (see Figure 2).

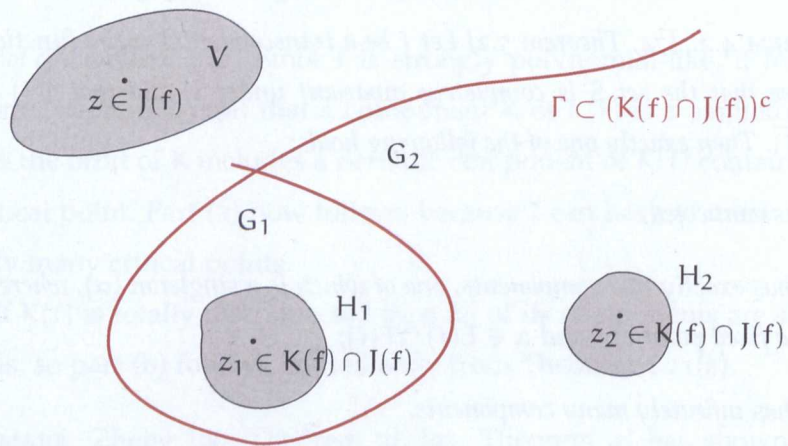


Figure 2: Proof of Theorem 1.10(a).

Suppose, then, that  $z_i \in G_i \cap K(f) \cap J(f)$  for  $i = 1, 2$ , and let  $H_i$  be a bounded open neighbourhood of  $z_i$  compactly contained in  $G_i$ . Since  $J(f)$  is perfect we may without loss of generality assume that neither  $\overline{H}_1$  nor  $\overline{H}_2$  meets  $E(f)$ .

Now let  $z$  be an arbitrary point in  $J(f)$ , and let  $V$  be a bounded open neighbourhood of  $z$ . Then, by the blowing up property of  $J(f)$  (Lemma 2.2(d)), there exists  $K \in \mathbb{N}$  such that

$$f^k(V) \supset \overline{H}_1 \cup \overline{H}_2 \quad (4.5)$$

for all  $k \geq K$ . Furthermore, there exists  $M \geq K$  such that

$$f^m(H_1) \supset \overline{H}_1 \cup \overline{H}_2 \quad \text{and} \quad f^m(H_2) \supset \overline{H}_1 \cup \overline{H}_2, \quad (4.6)$$

for all  $m \geq M$ .

Now let  $s = s_1 s_2 s_3 \dots$  be an infinite sequence of 1s and 2s. We show that each such sequence  $s$  can be associated with the orbit of a point in  $\bar{V} \cap K(f) \cap J(f)$ , as follows.

Put  $S_0 = \bar{V}$  and, for  $n \in \mathbb{N}$ , put  $S_n = \bar{H}_i$  if  $s_n = i$ . It follows from (4.5), (4.6) and Lemma 3.8 that there exists a point  $\zeta_s \in J(f)$  such that  $f^{Mn}(\zeta_s) \in S_n$  for  $n \geq 0$ . In particular,  $\zeta_s \in \bar{V}$ . Furthermore, for all  $k \geq 0$  we have

$$f^k(\zeta_s) \in \bigcup_{j=0}^{M-1} f^j(\bar{V}) \cup f^j(\bar{H}_1 \cup \bar{H}_2),$$

so  $\zeta_s$  has bounded orbit and thus lies in  $K(f)$ .

Now the points in  $\bar{V} \cap K(f) \cap J(f)$  whose orbits are associated with two different infinite sequences of 1s and 2s must lie in different components of  $K(f) \cap J(f)$ . For if two such sequences first differ in the  $N$ th term, then the  $MN$ th iterate of one point will lie in  $G_1$  and the other in  $G_2$ . Thus, if the two points were in the same component  $K$  of  $K(f) \cap J(f)$ , then  $f^{MN}(K)$  would meet  $\Gamma \subset (K(f) \cap J(f))^c$ , which is a contradiction.

Now there are uncountably many possible infinite sequences of 1s and 2s, so we have shown that every neighbourhood of an arbitrary point in  $J(f)$  meets uncountably many components of  $K(f) \cap J(f)$ , as required.

The proof of part (b) is similar, but we now make the additional assumption that  $f$  is strongly polynomial-like. Since we are assuming that  $K(f)$  is disconnected, it follows from Lemma 4.3 that there is a continuum in  $K(f)^c$  with two complementary components, each of which contains points in  $K(f)$ . As in the proof of part (a), we label the continuum  $\Gamma$  and the complementary components  $G_1$  and  $G_2$ .

We show that, in fact, each of  $G_1$  and  $G_2$  must contain points in  $K(f) \cap J(f)$ . For if not,  $G_i \subset F(f)$  for some  $i \in \{1, 2\}$ . However, since  $f$  is strongly polynomial-like, it follows from Theorem 1.11(b) that the Fatou component  $U$  containing  $G_i$  must be bounded and non-wandering, so that  $\bar{U} \subset K(f)$ . Thus  $\bar{U} \subset G_i$ , which is a contradiction.

*Components of  $K(f)$   
when  $f$  is strongly  
polynomial-like*

So, as before, we may choose  $z_i \in G_i \cap K(f) \cap J(f)$  for  $i = 1, 2$ , and bounded open neighbourhoods  $H_i$  of  $z_i$  compactly contained in  $G_i$ . The proof now proceeds exactly as for the proof of part (a), but we conclude that points in  $\bar{V} \cap K(f)$  whose orbits are associated with two different infinite sequences of 1s and 2s must lie in different components of  $K(f)$ . It follows that every neighbourhood of an arbitrary point in  $J(f)$  meets uncountably many components of  $K(f)$ .  $\square$

REMARKS. 1. We note that  $K(f) \cap J(f)$  can be connected, for example when  $f(z) = \sin z$ . For in proving the connectedness of  $J(f)$  in [28, Theorem 4.1], Domínguez also showed that the union  $E$  of the boundaries of all Fatou components is connected. Since, for this function, all Fatou components are bounded and  $F(f) \subset K(f)$ , it follows that  $E \subset K(f) \cap J(f) \subset J(f)$  and hence that  $K(f) \cap J(f)$  is connected. A similar argument shows that  $K(f)$  is connected.

2. We know of no example of a strongly polynomial-like function  $f$  for which  $K(f)$  is connected.
3. It follows from Theorem 1.10(b) and Corollary 1.12(a) that, if  $f$  is strongly polynomial-like and  $K(f)$  is disconnected, then  $K(f)$  has uncountably many singleton components.

#### 4.3 STRONGLY POLYNOMIAL-LIKE FUNCTIONS

In this section we prove Theorem 1.13, which gives a useful equivalent characterization of a strongly polynomial-like function, and Theorem 1.14, which gives several large classes of transcendental entire functions which are strongly polynomial-like.

*An equivalent  
characterization  
of strongly  
polynomial-like  
functions*

Recall that Theorem 1.13 says that a transcendental entire function  $f$  is strongly polynomial-like if and only if there exists a sequence of bounded, simply connected domains  $(D_n)_{n \in \mathbb{N}}$  such that

- $\bar{D}_n \subset D_{n+1}$ , for  $n \in \mathbb{N}$ ,
- $\bigcup_{n \in \mathbb{N}} D_n = \mathbb{C}$ , and
- $f(\partial D_n)$  surrounds  $\bar{D}_n$ , for  $n \in \mathbb{N}$ .



*Proof of Theorem 1.13.* First, suppose that  $f$  is strongly polynomial-like and let  $(V_n), (W_n)$  be the sequences of bounded, simply connected domains in Definition 1.9. Since  $(f; V_n, W_n)$  is a polynomial-like mapping, it follows that  $\bar{V}_n \subset W_n$  and  $f(\partial V_n) = \partial W_n$ , for  $n \in \mathbb{N}$ . Moreover, taking a subsequence of  $(V_n)_{n \in \mathbb{N}}$  if necessary, we can assume that  $W_n \subset V_{n+1}$  for  $n \in \mathbb{N}$ . Putting  $D_n = V_n$  for  $n \in \mathbb{N}$  then gives a sequence of domains with the properties stated in the theorem.

For the converse, let  $(D_n)_{n \in \mathbb{N}}$  be a sequence of bounded, simply connected domains with the properties stated in the theorem. Since  $f(D_n)$  is bounded, we may assume without loss of generality that

$$f(D_n) \subset D_{n+1}, \quad \text{for } n \in \mathbb{N}. \quad (4.7)$$

Now, for each  $n \in \mathbb{N}$ , let  $\Gamma_n$  be a smooth Jordan curve that surrounds  $\bar{D}_{n+1}$  and lies in the complementary component of  $f(\partial D_{n+1})$  containing  $\bar{D}_{n+1}$  (see Figure 3). Observe that it follows from the properties of the sequence  $(D_n)_{n \in \mathbb{N}}$  that  $f$  has no finite asymptotic values. Furthermore, we may assume that each  $\Gamma_n$  does not meet any of the critical values of  $f$ .

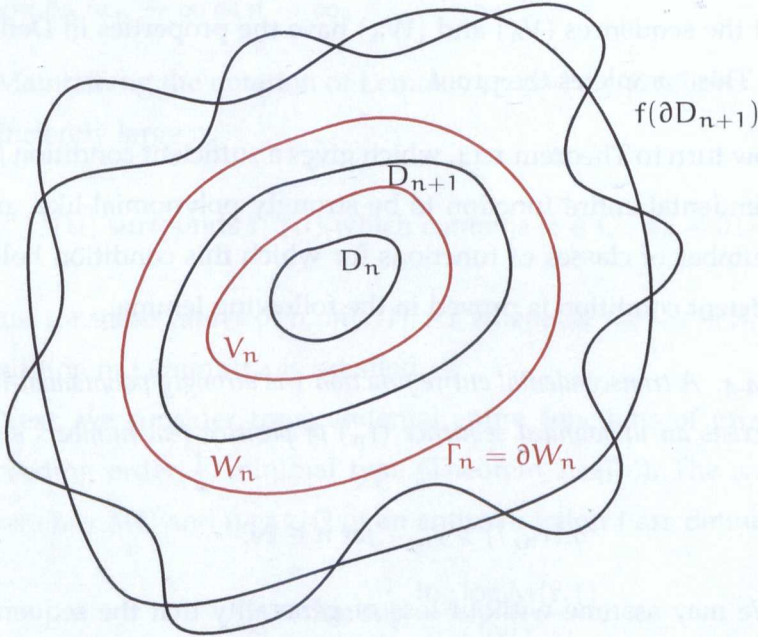


Figure 3: Proof of Theorem 1.13 - an alternative characterisation of strongly polynomial-like functions.

Let  $W_n$  denote the bounded complementary component of  $\Gamma_n$ . Then  $W_n$  contains  $D_{n+1}$  and hence  $f(D_n)$  by (4.7). Thus there is a component  $V_n$  of  $f^{-1}(W_n)$  that contains  $D_n$ . Furthermore,  $f : V_n \rightarrow W_n$  is a proper mapping, and since  $f$  is transcendental we may assume that the degree of this mapping is at least 2.

We claim that  $\overline{V_n} \subset W_n$ . For suppose not. Then since  $\partial D_{n+1} \subset W_n$  and  $D_n \subset V_n \cap D_{n+1}$  we must have  $V_n \cap \partial D_{n+1} \neq \emptyset$ . However, if  $\zeta \in V_n \cap \partial D_{n+1}$  then it follows that  $f(\zeta) \in W_n \cap f(\partial D_{n+1})$ , which contradicts the fact that  $W_n$  and  $f(\partial D_{n+1})$  are disjoint.

Moreover,  $V_n$  is simply connected. For suppose that  $V_n$  is multiply connected, and let  $\gamma$  be a Jordan curve in  $V_n$  which is not null homotopic there. Let  $G$  be the bounded complementary component of  $\gamma$ , so that  $G$  contains a component of  $\partial V_n$ . Now since  $f$  is a proper mapping we have  $f(\partial V_n) = \Gamma_n = \partial W_n$ , so  $f(G) \cap \Gamma_n \neq \emptyset$ , which is impossible because  $f(\gamma) \subset W_n$  and  $f(G)$  is bounded. Thus  $V_n$  is indeed simply connected, and since  $\Gamma_n$  meets no critical values of  $f$ ,  $\partial V_n$  is a smooth Jordan curve.

This establishes that, for each  $n \in \mathbb{N}$ , the triple  $(f; V_n, W_n)$  is a polynomial-like mapping. Furthermore, it follows from the construction that the sequences  $(V_n)$  and  $(W_n)$  have the properties in Definition 1.9. This completes the proof.  $\square$

We now turn to Theorem 1.14, which gives a sufficient condition for a transcendental entire function to be strongly polynomial-like, and lists a number of classes of functions for which this condition holds. The sufficient condition is proved in the following lemma.

*A sufficient condition for  $f$  to be strongly polynomial-like*

**LEMMA 4.4.** *A transcendental entire function  $f$  is strongly polynomial-like if there exists an unbounded sequence  $(r_n)$  of positive real numbers such that*

$$m(r_n, f) > r_n, \quad \text{for } n \in \mathbb{N}.$$

*Proof.* We may assume without loss of generality that the sequence  $(r_n)$  is strictly increasing. Putting  $D_n = \{z : |z| < r_n\}$ , we then have  $\overline{D_n} \subset D_{n+1}$ , for  $n \in \mathbb{N}$ , and  $\bigcup_{n \in \mathbb{N}} D_n = \mathbb{C}$ . Moreover, since a transcendental entire function always has points of period 2,  $f(\partial D_n)$  must

surround  $\overline{D}_n$  for sufficiently large  $n$ . The result now follows from Theorem 1.13.  $\square$

To complete the proof of Theorem 1.14, we discuss in turn each of the four classes of functions listed in the theorem and show that they meet the condition in Lemma 4.4.

First, we consider transcendental entire functions with a multiply connected Fatou component (Theorem 1.14(a)). Recall that we gave the basic properties of such components, proved by Baker, in Lemma 2.7. Later results have shown that the iterates of a multiply connected Fatou component eventually contain very large annuli. The following special case of a result of Zheng [92] is quoted in this form by Bergweiler, Rippon and Stallard in [21].

*Functions with a multiply connected Fatou component*

**LEMMA 4.5.** *Let  $f$  be a transcendental entire function with a multiply connected Fatou component  $U$ . If  $A \subset U$  is a domain containing a closed curve that is not null-homotopic in  $U$  then, for sufficiently large  $n \in \mathbb{N}$ ,*

$$f^n(U) \supset f^n(A) \supset \{z \in \mathbb{C} : \alpha_n < |z| < \beta_n\},$$

where  $\beta_n/\alpha_n \rightarrow \infty$  as  $n \rightarrow \infty$ .

Maintaining the notation of Lemmas 2.7 and 4.5, it follows that, for sufficiently large  $n$ ,

$$f^{n+1}(U) \text{ surrounds } f^n(U) \text{ which contains } \{z \in \mathbb{C} : \alpha_n < |z| < \beta_n\}.$$

Thus, for these values of  $n$ ,  $m(r, f) > r$  whenever  $\alpha_n < r < \beta_n$ , so the condition in Lemma 4.4 is satisfied.

Next, we consider transcendental entire functions of growth not exceeding order  $\frac{1}{2}$ , minimal type (Theorem 1.14(b)). The order  $\rho(f)$ , lower order  $\lambda(f)$  and type  $\tau(f)$  of an entire function  $f$  are defined by

*Functions of small growth*

$$\rho(f) := \limsup_{r \rightarrow \infty} \frac{\log \log M(r, f)}{\log r},$$

$$\lambda(f) := \liminf_{r \rightarrow \infty} \frac{\log \log M(r, f)}{\log r},$$

and

$$\tau(f) := \limsup_{r \rightarrow \infty} \frac{\log M(r, f)}{r^\rho}.$$

If  $\tau(f) = 0$ ,  $f$  is said to be of *minimal type*.

The following lemma implies Theorem 1.14(b) immediately.

LEMMA 4.6. *Let  $f$  be a transcendental entire function of growth not exceeding order  $\frac{1}{2}$ , minimal type, and let  $n \in \{0, 1, \dots\}$ . Then*

$$\limsup_{r \rightarrow \infty} \frac{m(r, f)}{r^n} = \infty.$$

This well-known result is proved for the case  $n = 0$  and  $\rho(f) < \frac{1}{2}$  in [87, p. 274]. The proof in the case of order  $\frac{1}{2}$ , minimal type, is similar, and the case  $n > 0$  follows by a standard argument; see, for example, [41, p.193].

Functions of finite  
order with Fabry  
gaps, or with the  
pits effect

Finally, we consider transcendental entire functions of finite order and with Fabry gaps (Theorem 1.14(c)) or which exhibit the pits effect in the sense defined by Littlewood and Offord (Theorem 1.14(d)).

A transcendental entire function  $f$  has *Fabry gaps* if

$$f(z) = \sum_{k=0}^{\infty} a_k z^{n_k},$$

where  $n_k/k \rightarrow \infty$  as  $k \rightarrow \infty$ . Loosely speaking, a function exhibits the *pits effect* if it has very large modulus except in small regions (pits) around its zeros. Littlewood and Offord [50] showed that, if  $\sum_{n=0}^{\infty} a_n z^n$  is a transcendental entire function of order  $\rho \in (0, \infty)$  and lower order  $\lambda > 0$ , and if

$$C = \left\{ f : f(z) = \sum_{n=0}^{\infty} \varepsilon_n a_n z^n \right\}$$

where the  $\varepsilon_n$  take the values  $\pm 1$  with equal probability, then almost all functions in the set  $C$  show the pits effect in a way made precise in [50]. For further discussion of the pits effect, we refer to [73, Section 8].



It is noted in [73, Section 8] that, if  $f$  has finite order and Fabry gaps, or if  $f \in C$  exhibits the pits effect in the sense defined by Littlewood and Offord, then for some  $p > 1$  and all sufficiently large  $r$ ,

$$\text{there exists } r' \in (r, r^p) \text{ with } m(r', f) \geq M(r, f). \quad (4.8)$$

It follows that, for these functions also, the condition in Lemma 4.4 is satisfied. This completes the proof of Theorem 1.14.

REMARK. It is also noted in [73, Section 8] that (4.8) holds for

- certain functions of infinite order which satisfy a suitable gap series condition, and
- functions other than those studied by Littlewood and Offord which have a suitably strong version of the pits effect.

Evidently, these functions also are strongly polynomial-like.

#### 4.4 EXAMPLES FOR WHICH $K(f)$ IS TOTALLY DISCONNECTED

In this section and the next we illustrate our results with a number of examples. The examples in this section are of transcendental entire functions for which  $K(f)$  is totally disconnected. In Section 4.5, we give an example of a transcendental entire function for which  $K(f)$  has a component with empty interior which is not a singleton.

EXAMPLE 4.7. Let  $f$  be the transcendental entire function constructed by Baker and Domínguez in [8, Theorem G]. Then  $K(f)$  is totally disconnected.

*$K(f)$  totally disconnected: a function of Baker and Domínguez*

*Proof.* The function  $f$  constructed in [8, Theorem G] takes the form

$$f(z) = k \prod_{n=1}^{\infty} \left(1 + \frac{z}{r_n}\right)^2, \quad 0 < r_1 < r_2 < \cdots, \quad k > 0,$$

where the constants  $k$  and  $r_n, n \in \mathbb{N}$ , are chosen so that  $f(x) > x$  for  $x \in \mathbb{R}$  and so that the annuli

$$A_n = \left\{ z : 2r_n^2 < |z| < \left(\frac{r_{n+1}}{2}\right)^{1/2} \right\}$$

are disjoint, with  $f(A_n) \subset A_{n+1}$  for large  $n$  (we refer to [8, proof of Theorem G] for details of the construction).

As noted in [8],  $f$  has order zero. Thus  $f$  is strongly polynomial-like, by Theorem 1.14(b). Furthermore, the construction ensures that  $f(x) > x$  for  $x \in \mathbb{R}$ , so it is easy to see that  $\mathbb{R} \subset I(f)$ . Since all critical points of  $f$  lie on the negative real axis, it follows that none are in  $K(f)$  and hence that  $K(f)$  is totally disconnected by Corollary 1.12(b).  $\square$

The function in Example 4.7 has multiply connected Fatou components. This fact gives an alternative method of showing that  $K(f)$  is totally disconnected by using results due to Kisaka [46] (see Section 3.5 for a discussion of these results). Recall that a *buried point* is a point in the Julia set that does not lie on the boundary of a Fatou component, and that a *buried component* of the Julia set is a component consisting entirely of buried points. In [46, Corollary D] Kisaka proved that, if a transcendental entire function has a multiply connected Fatou component and each critical point has an unbounded forward orbit, then every component of the Julia set with bounded orbit must be a buried singleton component. In [46, Example E], he showed that this result applies to the function  $f$  in Example 4.7. Since, for this function, no component of  $J(f)$  with bounded orbit meets the boundary of a Fatou component, it follows that  $K(f) \subset J(f)$  and hence that  $K(f)$  is totally disconnected.

In our next example,  $K(f)$  is again totally disconnected, but this time  $f$  has no multiply connected Fatou components.

$K(f)$  totally  
disconnected: no  
multiply connected  
Fatou components

EXAMPLE 4.8. Define  $f$  by

$$f(z) = \prod_{n=1}^{\infty} \left(1 + \frac{z}{2^n}\right)^2.$$

Then  $K(f)$  is totally disconnected. Moreover,  $f$  has no multiply connected Fatou components.

*Proof.* Since  $f$  is a canonical product with zeros at  $z = -2^n, n \in \mathbb{N}$ , and  $\sum_{n \in \mathbb{N}} 2^{-n\alpha}$  is convergent for all  $\alpha > 0$ , it follows that  $f$  has

order zero [87, p. 251]. Thus  $f$  is strongly polynomial-like by Theorem 1.14(b). Furthermore, for  $x \in \mathbb{R}$ ,

$$f(x) \geq \left(1 + \frac{x}{2}\right)^2 > x$$

so that  $\mathbb{R} \subset I(f)$ . Since all critical points of  $f$  lie on the negative real axis, it follows that none of them are in  $K(f)$ . Thus  $K(f)$  is totally disconnected by Corollary 1.12(b).

Now suppose that some component  $U$  of  $F(f)$  is multiply connected. Then, for large  $n$ , we have

$$f^{n+1}(U) \text{ surrounds } f^n(U) \text{ which surrounds } 0$$

by Lemma 2.7, so that  $f^n(U)$  contains no zeros of  $f$  for large  $n$ . However, by Lemma 4.5,  $f^n(U)$  contains an annulus  $\{z : \alpha_n < |z| < \beta_n\}$  for large  $n$ , where  $\beta_n/\alpha_n \rightarrow \infty$  as  $n \rightarrow \infty$ . Since the zeros of  $f$  are at  $z = -2^n, n \in \mathbb{N}$ , this is a contradiction and it follows that  $f$  has no multiply connected Fatou components.  $\square$

In Examples 4.7 and 4.8 the critical points of  $f$  lie outside  $K(f)$ . This is not essential for  $K(f)$  to be totally disconnected, and in our next example all of the critical points are inside  $K(f)$ .

**EXAMPLE 4.9.** Let  $f$  be the transcendental entire function constructed by Kisaka and Shishikura in [47, Theorem B]. Then  $K(f)$  is totally disconnected. Moreover, each critical point of  $f$  lies in a strictly preperiodic component of  $K(f)$ .

$K(f)$  totally  
disconnected: critical  
points in  $K(f)$

*Proof.* In [47, Theorem B], Kisaka and Shishikura used quasiconformal surgery to construct a transcendental entire function  $f$  with a doubly connected Fatou component which remains doubly connected throughout its orbit. It follows from Theorem 1.14(a) that  $f$  is strongly polynomial-like.

Now the construction of  $f$  in [47] ensures that all the critical values of  $f$  map to 0, which is a repelling fixed point. Furthermore, each critical value of  $f$  lies in the unbounded complementary component of at least one doubly connected Fatou component that surrounds 0. Thus the component  $K_0$  of  $K(f)$  containing 0 cannot include a critical

point, for if it did  $f(K_0)$  would meet a doubly connected Fatou component, which is a contradiction. Hence each critical point lies in a component of  $K(f)$  which differs from  $K_0$  and is strictly preperiodic. It follows from Corollary 1.12(b) that  $K(f)$  is totally disconnected.  $\square$

$A_R(f)$  is a spider's  
web for the  
above examples

Recall from the remarks in the introduction to this chapter that a transcendental entire function  $f$  is strongly polynomial-like whenever the set  $A_R(f)$  is a spider's web. In fact,  $A_R(f)$  is a spider's web for the functions in each of the above examples, as we now show. We use the following result due to Rippon and Stallard.

**THEOREM 4.10.** [73, part of Theorem 1.9] *Let  $f$  be a transcendental entire function and let  $R > 0$  be such that  $M(r) > r$  for  $r \geq R$ . Then  $A_R(f)$  is a spider's web if either of the following holds:*

- (a)  *$f$  has a multiply connected Fatou component;*
- (b)  *$f$  has very small growth; that is, there exist  $m \geq 2$  and  $r_0 > 0$  such that*

$$\log \log M(r) < \frac{\log r}{\log^m r}, \text{ for } r > r_0,$$

*where  $\log^m$  denotes the  $m$ th iterated logarithm function.*

It is immediate from Theorem 4.10(a) that  $A_R(f)$  is a spider's web for the functions in Examples 4.7 and 4.9. To show that  $A_R(f)$  is also a spider's web for the function in Example 4.8, let  $r \geq 1$  and let  $N \in \mathbb{N}$  be such that  $2^{N-1} \leq r < 2^N$ . Then since  $M(r) = f(r)$ , we have

$$\begin{aligned} \log M(r) &= 2 \sum_{n=1}^{\infty} \log \left( 1 + \frac{r}{2^n} \right) \\ &= 2 \sum_{n=1}^N \log \left( 1 + \frac{r}{2^n} \right) + 2 \sum_{n=N+1}^{\infty} \log \left( 1 + \frac{r}{2^n} \right) \\ &< 2 \sum_{n=1}^N \log \frac{2^N}{2^{n-1}} + 2 \sum_{n=N+1}^{\infty} \frac{2^N}{2^n} \\ &< 2N^2 \log 2 + 2. \end{aligned}$$

Now  $N \leq 1 + \frac{\log r}{\log 2}$ , so for sufficiently large  $r$  there exists  $C > 0$  such that

$$\log M(r) < C(\log r)^2.$$

However, this means that

$$\frac{\log \log M(r)}{\log r} < C' \frac{\log \log r}{\log r} < \frac{1}{\log \log r}$$

for some  $C' > 0$  and for sufficiently large  $r$ . The fact that  $A_R(f)$  is a spider's web now follows from Theorem 4.10(b) with  $m = 2$ .

For our final example in this section, we exhibit a transcendental entire function which is not strongly polynomial-like, but for which  $K(f)$  is totally disconnected.

$K(f)$  totally  
disconnected:  
 $f$  not strongly  
polynomial-like

EXAMPLE 4.11. Let  $f$  be the function

$$f(z) = z + 1 + e^{-z},$$

first investigated by Fatou [37]. Then  $K(f)$  is totally disconnected.

*Proof.* Fatou [37, Example 1] demonstrated that  $F(f)$  is a completely invariant Baker domain in which  $f^n(z) \rightarrow \infty$  as  $n \rightarrow \infty$ . As stated in [72, Example 3], it can be shown using a result of Barański [11, Theorem C], together with the fact that  $f$  is the lift of  $g(w) = (1/e)we^{-w}$  under  $w = e^{-z}$ , that:

- $J(f)$  consists of uncountably many disjoint simple curves, each with one finite endpoint and the other endpoint at  $\infty$ , and
- $I(f) \cap J(f)$  consists of the open curves and some of their finite endpoints.

Thus all points in  $F(f)$  and all points on the curves to infinity in  $J(f)$ , together with some of their finite endpoints, lie in the escaping set  $I(f)$ . It follows that  $K(f)$  is a subset of the finite endpoints of the curves to infinity in  $J(f)$ . Thus, if the set of finite endpoints of these curves is totally disconnected, then  $K(f)$  is totally disconnected.

Now it follows from [13, Theorem 1.5] that  $J(f)$  is a Cantor bouquet, in the sense of being ambiently homeomorphic to a subset of

$\mathbb{R}^2$  known as a straight brush (we refer to [1, 13] for a detailed discussion of these ideas). Now Mayer [51, Theorem 3] has shown that, if  $h(z) = \lambda e^z$ ,  $0 < \lambda < 1/e$ , the set of finite endpoints of  $J(h)$  is totally disconnected. Since  $J(h)$  is also a Cantor bouquet, it is ambiently homeomorphic to  $J(f)$ . We conclude that the set of finite endpoints of  $J(f)$  is totally disconnected, and this completes the proof.  $\square$

#### 4.5 A NON-TRIVIAL COMPONENT OF $K(f)$ WITH EMPTY INTERIOR

In this final section of the present chapter, we use quasiconformal surgery to construct a transcendental entire function for which  $K(f)$  has a component with no interior that is not a singleton. A brief introduction to quasiconformal mappings and the idea of quasiconformal surgery is given in Section 2.3.

*The technique of  
quasiconformal  
surgery*

We obtain a transcendental entire function with the desired dynamical behaviour by modifying a construction of Bergweiler [17], which is itself based on an approach used by Kisaka and Shishikura in [47] (for which see also Example 4.9). The main theorem used in these results, which yields the required entire function from quasiregular mappings constructed to have similar dynamical behaviour, is stated in Section 2.3 (Theorem 2.10).

We will also need the following lemma. Here,  $\log$  denotes the principal branch of the logarithm.

**LEMMA 4.12.** [47, Lemma 6.2] *Let  $k \in \mathbb{N}$ ,  $0 < r_1 < r_2$ , and for  $j = 1, 2$ , let  $\phi_j$  be analytic on a neighbourhood of  $\{z : |z| = r_j\}$  and such that  $\phi_j|_{|z|=r_j}$  goes round the origin  $k$  times. If*

$$\left| \log \left( \frac{\phi_2(r_2 e^{iy})}{r_2^k} \frac{r_1^k}{\phi_1(r_1 e^{iy})} \right) \right| \leq \delta_0 \quad (4.9)$$

and

$$\left| z \frac{d}{dz} \left( \log \frac{\phi_j(z)}{z^k} \right) \right| \leq \delta_1, \quad z = r_j e^{iy}, \quad j = 1, 2, \quad (4.10)$$

hold for every  $y \in (-\pi, \pi]$  and for some positive constants  $\delta_0$  and  $\delta_1$  satisfying

$$C = 1 - \frac{1}{k} \left( \frac{\delta_0}{\log(r_2/r_1)} + \delta_1 \right) > 0, \quad (4.11)$$

then there exists a  $K$ -quasiregular mapping

$$H : \{z : r_1 \leq |z| \leq r_2\} \rightarrow \mathbb{C} \setminus \{0\}$$

with  $K \leq 1/C$ , such that  $H$  has no critical points and

$$H = \phi_j \text{ on } \{z : |z| = r_j\}, \quad j = 1, 2.$$

We now give the details of the construction of a transcendental entire function with the desired property.

**EXAMPLE 4.13.** There exists a transcendental entire function  $f$  such that  $K(f)$  has a component which has empty interior but which is not a singleton.

*Proof.* We first define a quasiregular mapping  $g$  and then obtain the required entire function  $f$  using Theorem 2.10.

In Bergweiler's construction [17], sequences  $(a_n)$  and  $(R_n)$  are chosen so that  $z \mapsto a_n z^{n+1}$  maps  $\text{ann}(R_n, R_{n+1})$  onto  $\text{ann}(R_{n+1}, R_{n+2})$ , where

*Details of  
Bergweiler's  
construction*

$$\text{ann}(r_1, r_2) := \{z \in \mathbb{C} : r_1 < |z| < r_2\}, \quad r_2 > r_1 > 0.$$

The mapping  $g$  is then defined by  $g(z) = a_n z^{n+1}$  on a large subannulus of  $\text{ann}(R_n, R_{n+1})$  for each  $n \in \mathbb{N}$ , and by interpolation using [47, Lemma 6.3] (see also [17, Lemma 2]) in the annuli containing the circles  $\{z : |z| = R_n\}$  that lie between these subannuli. We modify Bergweiler's construction only on a disc surrounding the origin.

First we define the boundaries of the various annuli we will need. Here we follow Bergweiler precisely but we give the details for convenience. Set  $R_0 = 1$ . Choose  $R_1 > R_0$  and put

$$R_{n+1} := \frac{R_n^{n+1}}{R_{n-1}^n}$$

for  $n \in \mathbb{N}$ . With  $\gamma = \log R_1$  we then have

$$\log \frac{R_{n+1}}{R_n} = n \log \frac{R_n}{R_{n-1}} = \dots = n! \log \frac{R_1}{R_0} = \gamma n!.$$

Now define sequences  $(P_n)$ ,  $(Q_n)$ ,  $(S_n)$  and  $(T_n)$  by

$$\log \frac{T_n}{S_n} = \log \frac{S_n}{R_n} = \log \frac{R_n}{Q_n} = \log \frac{Q_n}{P_n} = \sqrt{\log \frac{R_{n+1}}{R_n}} = \sqrt{\gamma n!}. \quad (4.12)$$

Setting  $R_1 > e$  gives  $\gamma > 1$  and so

$$\frac{T_n}{S_n} = \frac{S_n}{R_n} = \frac{R_n}{Q_n} = \frac{Q_n}{P_n} > e.$$

We also have

$$\begin{aligned} \log \frac{P_{n+1}}{T_n} &= -\log \frac{Q_{n+1}}{P_{n+1}} - \log \frac{R_{n+1}}{Q_{n+1}} + \log \frac{R_{n+1}}{R_n} - \log \frac{S_n}{R_n} - \log \frac{T_n}{S_n} \\ &= -2\sqrt{\gamma(n+1)!} + \gamma n! - 2\sqrt{\gamma n!} > 0, \end{aligned}$$

provided  $R_1$  and hence  $\gamma$  is sufficiently large. It follows that

$$P_n < Q_n < R_n < S_n < T_n < P_{n+1}$$

for all  $n \in \mathbb{N}$ .

Now, again following Bergweiler, define sequences  $(a_n)$  and  $(b_n)$  as follows:

$$a_n := \frac{R_{n+1}}{R_n^{n+1}} = \frac{1}{R_{n-1}^n},$$

and

$$b_n := -\frac{(n+1)^2}{n+2} \left( \frac{n+1}{n} \right)^n a_n.$$

*Properties of the  
quasiregular  
mapping  $g$*

We will show that there is a quasiregular mapping  $g : \mathbb{C} \rightarrow \mathbb{C}$  with the following properties:

- (i)  $g(z) = z^2 - 2$  for  $|z| \leq S_1$ ;
- (ii)  $g(z) = a_n z^{n+1}$  for  $T_n \leq |z| \leq P_{n+1}$ ,  $n \geq 1$ ;
- (iii)  $g(z) = b_n(z - R_n)z^n$  for  $Q_n \leq |z| \leq S_n$ ,  $n \geq 2$ ;



(iv)  $g$  is  $K_n$ -quasiregular in  $E_n$  for  $n \geq 1$ , where

$$E_n = \text{ann}(S_n, T_n) \cup \text{ann}(P_{n+1}, Q_{n+1}) \quad \text{and} \quad K_n = 1 + \frac{1}{n^2};$$

(v)  $g(\text{ann}(S_n, Q_{n+1})) \subset \text{ann}(S_{n+1}, Q_{n+2})$  for  $n \geq 1$ .

Our mapping  $g$  differs from the quasiregular mapping constructed by Bergweiler in [17] only in the disc  $\{z : |z| \leq P_2\}$ . Bergweiler's mapping was set equal to  $z^2$  throughout this disc (since  $a_1 = 1$ ), whereas our mapping is equal to  $z^2$  only in the closure of  $\text{ann}(T_1, P_2)$  and we have introduced the new function  $z^2 - 2$  in the smaller disc  $\{z : |z| \leq S_1\}$ . Thus Bergweiler's proof that his mapping has the stated properties applies without amendment to our mapping  $g$ , but we need to carry out an additional interpolation between the functions  $z^2 - 2$  and  $z^2$  in order to define  $g$  in  $\text{ann}(S_1, T_1)$ . We also need to check that property (v) still holds for  $n = 1$ .

To define  $g$  in  $\text{ann}(S_1, T_1)$  we apply Lemma 4.12 with

$$\phi_1(z) = z^2 - 2, \quad \phi_2(z) = z^2, \quad r_1 = S_1 \text{ and } r_2 = T_1.$$

Defining  $g$  in  
 $\text{ann}(S_1, T_1)$

Evidently  $k = 2$  in Lemma 4.12, so (4.9) becomes

$$\left| \log \left( \frac{T_1^2 e^{2iy}}{T_1^2} \frac{S_1^2}{S_1^2 e^{2iy} - 2} \right) \right| = \left| \log \left( 1 - \frac{2}{S_1^2} e^{-2iy} \right) \right|.$$

Now as  $y$  runs through the interval  $(-\pi, \pi]$ , the point  $z = 1 - \frac{2}{S_1^2} e^{-2iy}$  traces out a small circle with centre 1 (note that  $S_1 > eR_1 > e^2$ ). Thus for such  $z$  we have

$$|z| \leq 1 + \frac{2}{S_1^2}$$

and

$$|\arg z| \leq \sin^{-1} \frac{2}{S_1^2} \leq \frac{\pi}{S_1^2},$$

so  $\log |z| < \frac{2}{S_1^2}$  and

$$|\log z| < \sqrt{\frac{4}{S_1^4} + \frac{\pi^2}{S_1^4}} < \frac{4}{e^4}.$$

It follows that (4.9) is satisfied with  $\delta_0 = \frac{4}{e^4}$ .

Moreover for  $j = 1$ , (4.10) becomes

$$\left| z \frac{d}{dz} \left( \log \frac{z^2 - 2}{z^2} \right) \right| = \frac{4}{|z^2 - 2|},$$

where  $z = S_1 e^{iy}$ . But

$$\frac{4}{|z^2 - 2|} \leq \frac{4}{S_1^2 - 2} < \frac{4}{e^4 - 2},$$

so that (4.10) is satisfied with  $\delta_1 = \frac{4}{e^4 - 2}$ . For  $j = 2$ , (4.10) is satisfied for any  $\delta_1 > 0$ .

With these values of  $\delta_0$  and  $\delta_1$ , (4.11) gives

$$C = 1 - \frac{1}{2} \left( \frac{4}{e^4 \log(T_1/S_1)} + \frac{4}{e^4 - 2} \right) > \frac{1}{2}.$$

It follows that there exists a  $K$ -quasiregular mapping

$$H : \{z : S_1 \leq |z| \leq T_1\} \rightarrow \mathbb{C} \setminus \{0\}$$

with  $K \leq 2$ , such that  $H$  has no critical points,  $H(z) = z^2 - 2$  on  $\{z : |z| = S_1\}$  and  $H(z) = z^2$  on  $\{z : |z| = T_1\}$ . Thus, putting  $g(z) = H(z)$  in  $\text{ann}(S_1, T_1)$  we see that (iv) holds for all  $z \in E_1$ , since our definition of  $g$  coincides with Bergweiler's on  $\text{ann}(P_2, Q_2)$ .

*Checking that  
property (v) holds*

Next, we check that (v) still holds for  $z \in \text{ann}(S_1, Q_2)$ . Since our quasiregular mapping  $g$  agrees with Bergweiler's on  $\{z : |z| = Q_2\}$ , his argument that  $|g(z)| \leq Q_3$  for  $z \in \text{ann}(S_1, Q_2)$  (which uses the maximum principle) continues to hold. It therefore remains to show that, for such  $z$ , we have  $|g(z)| \geq S_2$ .

Now  $g$  has no zeros in  $\text{ann}(S_1, Q_2)$  so if  $z \in \text{ann}(S_1, Q_2)$  we have

$$|g(z)| \geq \min_{|z|=S_1} |z^2 - 2| \geq S_1^2 - 2,$$

by the minimum principle. Moreover, since  $R_1 = e^\gamma$  it follows that  $S_1 = R_1 e^{\sqrt{\gamma}} = e^{\gamma + \sqrt{\gamma}}$  by (4.12), and therefore

$$|g(z)| \geq e^{2\gamma + 2\sqrt{\gamma}} - 2$$

for  $z \in \text{ann}(S_1, Q_2)$ . Now

$$\log \frac{S_2}{R_1} = \log \frac{R_2}{R_1} + \log \frac{S_2}{R_2} = \gamma + \sqrt{2\gamma},$$

so that  $S_2 = R_1 e^{\gamma + \sqrt{2\gamma}} = e^{2\gamma + \sqrt{2\gamma}}$ . It follows that we can ensure that  $|g(z)| > S_2$  for  $z \in \text{ann}(S_1, Q_2)$  by choosing  $\gamma$  sufficiently large, and (v) will then still hold.

Our mapping  $g$  and the sets  $E_j, j \in \mathbb{N}$ , therefore meet the conditions of Theorem 2.10, and we conclude that there exists a  $K_\infty$ -quasiconformal mapping  $\phi : \mathbb{C} \rightarrow \mathbb{C}$  such that  $f = \phi \circ g \circ \phi^{-1}$  is an entire function. Now it follows from (v) that  $g^n(z) \rightarrow \infty$  as  $n \rightarrow \infty$  for  $z \in \text{ann}(S_1, Q_2)$ . However, inside the disc  $\{z : |z| \leq S_1\}$  the iterates of  $g$  are the iterates of  $z^2 - 2$ . In particular, the interval  $[-2, 2]$  is invariant under iteration by  $g$  and contains the critical point 0, whilst for all  $z \in \{z : |z| \leq S_1\} \setminus [-2, 2]$  there must be some  $N \in \mathbb{N}$  such that  $|g^N(z)| > S_1$ .

It follows that  $\phi(\text{ann}(S_1, Q_2))$  lies in a multiply connected component  $U$  of  $F(f)$ , whilst  $\phi([-2, 2])$  is an invariant Jordan arc which is a subset of a component  $K$  of  $K(f)$  containing a critical point. Now suppose that  $K$  contains some point  $w \notin \phi([-2, 2])$ . Then there exists  $N \in \mathbb{N}$  such that  $f^N(w)$  lies outside the image under  $\phi$  of the disc  $\{z : |z| \leq S_1\}$ . However, as  $f^N(K)$  is connected, this means that  $f^N(K)$  meets  $U$ , which is a contradiction since  $U \subset I(f)$  by Lemma 2.7. Thus  $K$  is a component of  $K(f)$  with empty interior. This completes the proof.  $\square$

REMARKS. 1. It follows from Theorem 1.2(c) that every neighbourhood of  $K$  contains a multiply connected Fatou component that surrounds  $K$ , and that  $K$  is a buried component of  $J(f)$ . Since  $f$  is strongly polynomial-like, there are at most countably many components of  $K(f)$  with empty interior that are not singletons by Corollary 1.12(a).

2. As we have modified Bergweiler's construction only inside the disc  $\{z : |z| \leq P_2\}$ , the conclusions of [17] still hold, and  $f$  has both simply and multiply connected wandering domains.

*An entire function  $f$   
with the desired  
properties*

## SPIDERS' WEBS AND LOCALLY CONNECTED JULIA SETS

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In this chapter, we give the proofs of our results on the links between the spider's web form of the Julia set observed for certain transcendental entire functions and the local connectedness of the Julia set. Our main result is that, if  $J(f)$  is locally connected for a transcendental entire function  $f$ , then  $J(f)$  is a spider's web (Theorem 1.15). In the opposite direction, we show that, if  $J(f)$  is a spider's web, then there is a dense subset of buried points at which  $J(f)$  is locally connected (Theorem 1.17).

Section 5.1 is devoted to some preliminaries. Then, in Section 5.2, we prove Theorem 1.15 and related results, whilst in Section 5.3 we prove Theorems 1.17 and 1.18. In Section 5.4, we give some results on the residual Julia set  $J_r(f)$ , including the fact that there are classes of functions for which  $J_r(f)$  is itself a spider's web. Finally, in Section 5.5, we give a number of examples to illustrate the results of previous sections. We show that the Julia set of the function  $\sin z$  is a spider's web, and give some new examples of transcendental entire functions for which the Julia set is locally connected.

*Organisation of  
the chapter*

### 5.1 PRELIMINARIES

In this brief section we recall the definition of local connectedness and state some results we will use in subsequent sections.

A Hausdorff space  $X$  is *locally connected* at the point  $x \in X$  if  $x$  has arbitrarily small connected (but not necessarily open) neighbourhoods in  $X$ . If this is true for every  $x \in X$ , then we say that  $X$  is locally connected (see, for example, Milnor [56, p. 182]).

*Definition of local  
connectedness and  
some topological  
preliminaries*

We will need the following topological results due to Whyburn. Here a *plane continuum* is a compact, connected set lying in the plane or on the Riemann sphere.

LEMMA 5.1. [88, Ch.VI, (4.4)] *A plane continuum is locally connected if and only if*

- (a) *the boundary of each of its complementary components is locally connected, and*
- (b) *for each  $\varepsilon > 0$ , at most finitely many of these complementary components have spherical diameters greater than  $\varepsilon$ .*

LEMMA 5.2. [88, Ch.VI, (4.5)] *If a point  $p$  in a locally connected plane continuum  $E$  is not on the boundary of any complementary component of  $E$ , then for each  $\varepsilon > 0$ ,  $E$  contains a Jordan curve of spherical diameter less than  $\varepsilon$  surrounding  $p$ .*

Connectedness  
properties of  $J(f)$

We also make use of the following results on the connectedness properties of the Julia set of a transcendental entire function, due to Kisaka and to Baker and Domínguez.

LEMMA 5.3. [44, Theorem 2] *If  $f$  is a transcendental entire function such that all components of  $F(f)$  are bounded and simply connected, then  $J(f)$  is connected.*

LEMMA 5.4. [8, part of Theorem A] *If  $f$  is a transcendental entire function such that  $J(f)$  is locally connected at one of its points, then  $J(f)$  is connected.*

LEMMA 5.5. [8, Corollary 3] *If  $f$  is a transcendental entire function and  $F(f)$  has a completely invariant component, then  $J(f)$  is not locally connected at any point.*

## 5.2 PROOF OF THEOREM 1.15 AND RELATED RESULTS

We now prove the following result and show that this implies Theorem 1.15.

Recall that, if  $U$  is a Fatou component such that  $f^{-1}(U) \subset U$ , then it follows that  $f(U) \subset U$ , and  $U$  is referred to as *completely invariant*. It

is shown in [5] that, if  $f$  is a transcendental entire function, there can be at most one such component.

**THEOREM 5.6.** *Let  $f$  be a transcendental entire function such that:*

- (1)  $F(f)$  has no completely invariant component, and
- (2) for each  $\varepsilon > 0$ , at most finitely many components of  $F(f)$  have spherical diameters greater than  $\varepsilon$ .

*Conditions under which all Fatou components are bounded and  $J(f)$  is a spider's web*

*Then  $F(f)$  has no unbounded components, and there is a sequence  $(G_k)_{k \in \mathbb{N}}$  of bounded, simply connected domains such that*

- $G_{k+1} \supset G_k$ , for  $k \in \mathbb{N}$ ,
- $\partial G_k \subset J(f)$ , for  $k \in \mathbb{N}$  and
- $\bigcup_{k \in \mathbb{N}} G_k = \mathbb{C}$ .

**COROLLARY 5.7.** *Let  $f$  be a transcendental entire function satisfying the assumptions of Theorem 5.6, and assume further that  $F(f)$  has no multiply connected components. Then  $J(f)$  is a spider's web.*

Note that, if  $J(f)$  is locally connected, it follows from Lemmas 5.1 and 5.5 that the assumptions of Theorem 5.6 hold. Theorem 1.15 then follows because  $J(f)$  is connected by Lemma 5.4.

We will need the following simple lemma, in which by a *preimage* of a Fatou component  $U$ , we mean a component of  $f^{-n}(U)$  for some  $n \in \mathbb{N}$ . This result is surely known, but we include a proof for completeness as we have been unable to locate a reference.

**LEMMA 5.8.** *Let  $f$  be a transcendental entire function. Then every component of  $F(f)$  which is not completely invariant has infinitely many distinct preimages.*

*Components of  $F(f)$  with infinitely many distinct preimages*

*Proof.* Assume, for a contradiction, that  $U$  is a component of  $F(f)$  which is not completely invariant and has only finitely many distinct preimages.

We first show  $U$  must be periodic. For suppose that  $U$  is non-periodic, and let  $U'$  be any preimage of  $U$ . Then  $U'$  is a component of  $f^{-n}(U)$  for some  $n \in \mathbb{N}$ , and since  $U$  is not periodic, there must be

at least one component of  $f^{-1}(U')$  which is distinct from every component of  $f^{-k}(U)$  for  $k = 1, \dots, n$ . As this is true for all components of  $f^{-n}(U)$  and for every  $n \in \mathbb{N}$ ,  $U$  must have infinitely many distinct preimages, which is against our assumption.

Thus  $U$  must belong to some cycle of period  $p > 1$  in which no element in the cycle has preimages outside the cycle. But then each of the  $p$  distinct elements in the cycle is a completely invariant component of  $F(f^p)$ , contradicting the fact that a transcendental entire function can have at most one completely invariant Fatou component [5]. This contradiction completes the proof.  $\square$

*Proof of conditions  
under which all  
Fatou components  
are bounded and  
 $J(f)$  is a spider's web*

*Proof of Theorem 5.6.* We first suppose that  $F(f)$  has an unbounded component  $V$ , and seek a contradiction.

Let  $R > 0$  be so large that  $B(0, R) \cap J(f) \neq \emptyset$ . Then we claim that infinitely many preimages of  $V$  must meet  $B(0, R)$ .

To show this, note that  $V$  has infinitely many distinct unbounded preimages, by Lemma 5.8. Thus, if only finitely many of these preimages meet  $B(0, R)$ , there must be some preimage  $W$  of  $V$  that is not periodic and does not meet  $B(0, R)$ . But since  $B(0, R) \cap J(f) \neq \emptyset$ , it follows from the blowing up property of  $J(f)$  (Lemma 2.2(d)) that there exists  $N \in \mathbb{N}$  such that  $f^n(B(0, R))$  meets  $W$  for all  $n \geq N$ . Thus we may choose a strictly increasing sequence  $(n_j)_{j \in \mathbb{N}}$  of integers greater than  $N$  such that some component  $X_{n_j}$  of  $f^{-n_j}(W)$  meets  $B(0, R)$  for all  $j \in \mathbb{N}$ .

Now suppose that two such components coincide. Then there exist  $k, l \in \mathbb{N}$  with  $k > l$  (and thus  $n_k > n_l$ ), and  $X_{n_k} = X_{n_l} = X$ , say, such that

$$f^{n_l}(X) \subset W$$

and

$$f^{n_k}(X) \subset W.$$

It then follows that  $f^{n_k - n_l}(W) \subset W$ , so that  $W$  is periodic, contrary to our assumption. This proves the claim.

Now since every point on the circle  $C(0, R)$  lies at the same spherical distance from  $\infty$ , the fact that infinitely many unbounded preimages of  $V$  meet  $B(0, R)$  contradicts property (2) in the statement of

Theorem 5.6. Thus it follows that there are no unbounded components of  $F(f)$ .

Now let  $r > 0$ , and let  $\varepsilon > 0$  be less than the spherical distance of the circle  $C = C(0, r)$  from  $\infty$  (see Figure 4). Let  $\{U_j : j \in \mathbb{N}\}$  be the collection of components of  $F(f)$  that meet  $C$ . This collection may be empty, finite or countably infinite, but

- (i) we have just proved that none of the  $U_j$  is unbounded, and
- (ii) by property (2) in the statement of the theorem, at most finitely many of the  $U_j$  have spherical diameters greater than  $\varepsilon$ .

It follows that  $\bigcup_{j \in \mathbb{N}} \bar{U}_j$  must be bounded. If we now let

$$D = C \cup \bigcup_{j \in \mathbb{N}} \bar{U}_j,$$

and put

$$G = \text{int}(\tilde{D}),$$

where  $\tilde{D}$  is the union of  $D$  and its bounded complementary components, we then have that  $G$  is a bounded, simply connected domain whose boundary  $\partial G$  lies in  $J(f)$ .

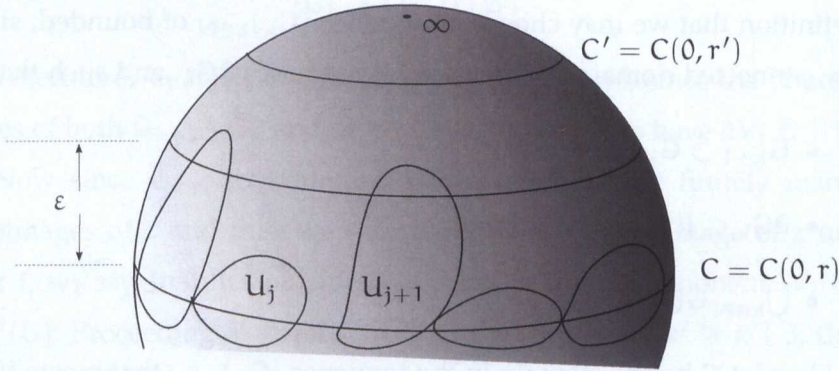


Figure 4: Construction of the loops in  $J(f)$  in the proof of Theorem 5.6, shown on the Riemann sphere.

Now choose  $r' > r$  such that  $G \subset B(0, r')$ , and let  $\varepsilon' > 0$  be less than the spherical distance of the circle  $C' = C(0, r')$  from  $\infty$ . Then we may proceed exactly as above to obtain a bounded, simply connected domain  $G' \supset G$  whose boundary  $\partial G'$  lies in  $J(f)$ .



In this way, we may construct a sequence  $(G_k)_{k \in \mathbb{N}}$  of bounded, simply connected domains such that  $G_{k+1} \supset G_k$  and  $\partial G_k \subset J(f)$  for each  $k \in \mathbb{N}$ , and  $\bigcup_{k \in \mathbb{N}} G_k = \mathbb{C}$ . This completes the proof.  $\square$

*Proof of Corollary 5.7.* To prove that  $J(f)$  is a spider's web, it only remains to show that  $J(f)$  is connected. But since there are no multiply connected Fatou components, this is immediate from Lemma 5.3.  $\square$

### 5.3 PROOF OF THEOREMS 1.17 AND 1.18

In this section we first prove Theorem 1.17, which says that if  $f$  is a transcendental entire function such that  $J(f)$  is a spider's web, then there exists a subset of  $J(f)$  which is dense in  $J(f)$  and consists of points  $z$  with the property that any neighbourhood of  $z$  contains a continuum in  $J(f)$  surrounding  $z$  and, furthermore, that each such point  $z$  is a buried point of  $J(f)$  at which  $J(f)$  is locally connected. The method of proof is similar to that adopted by Bergweiler [16] in his alternative proof of a result due to Domínguez [28] (compare also the proof of Theorem 1.7 in Section 3.6).

*Proof of Theorem  
1.17: properties of  
 $J(f)$  spider's webs*

*Proof of Theorem 1.17.* Since  $J(f)$  is a spider's web, it follows from the definition that we may choose a sequence  $(G_k)_{k \in \mathbb{N}}$  of bounded, simply connected domains with *disjoint* boundaries  $\partial G_k$ , and such that

- $G_{k+1} \supset G_k$ , for  $k \in \mathbb{N}$ ,
- $\partial G_k \subset J(f)$ , for  $k \in \mathbb{N}$  and
- $\bigcup_{k \in \mathbb{N}} G_k = \mathbb{C}$ .

Now let  $G$  be any domain in the sequence  $(G_k)_{k \in \mathbb{N}}$  that meets  $J(f)$ , and let  $z \in G \cap J(f)$  be such that  $z \notin E(f)$ . Then, by Picard's theorem, there are infinitely many preimages of  $z$  under  $f$ , and each of these must lie in a component of  $f^{-1}(G)$ . Note that a component of  $f^{-1}(G)$  can in general be either bounded or unbounded, and can contain more than one preimage of  $z$  under  $f$ .

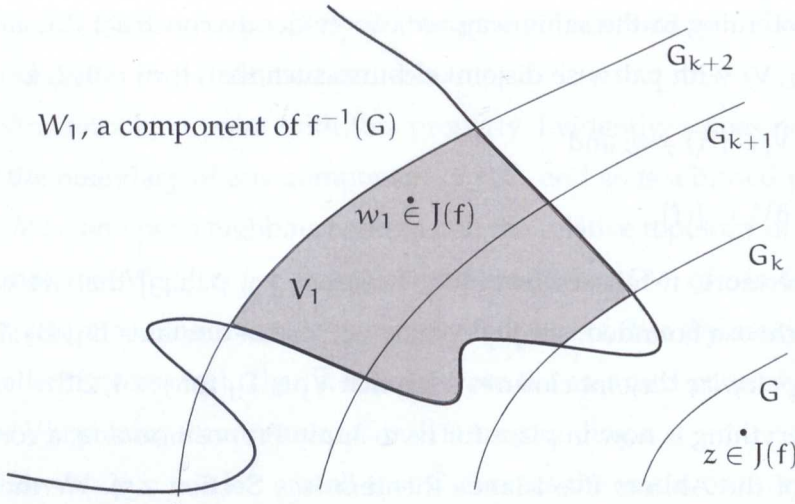


Figure 5: Construction of the domain  $V_1$  in the proof of Theorem 1.17.

Let  $w_1$  be some preimage of  $z$  under  $f$ , and let  $W_1$  be the component of  $f^{-1}(G)$  containing  $w_1$  (see Figure 5). Then we can assume that  $w_1$  is so large that there exists  $k \in \mathbb{N}$  such that

$$w_1 \in G_{k+2} \setminus \overline{G}_k.$$

It follows that  $w_1$  lies in a bounded domain  $V_1$  which is a component of

$$(G_{k+2} \setminus \overline{G}_k) \cap W_1.$$

Furthermore,  $V_1 \cap J(f) \neq \emptyset$  (because  $w_1 \in J(f)$ ) and, since the boundaries of both  $G_{k+2} \setminus \overline{G}_k$  and of  $W_1$  lie in  $J(f)$ , we also have  $\partial V_1 \subset J(f)$ .

Now since  $\overline{G}_{k+3}$  is bounded, it can contain only finitely many preimages of  $z$  and thus we may choose another preimage of  $z$  under  $f$ ,  $w_2$  say, that lies outside  $\overline{G}_{k+3}$  and in some component  $W_2$  of  $f^{-1}(G)$ . Proceeding as before, we find that, for some  $k' \geq k+3$ , the point  $w_2$  lies in a bounded domain  $V_2$  which is a component of

$$(G_{k'+2} \setminus \overline{G}_{k'}) \cap W_2.$$

We also have  $V_2 \cap J(f) \neq \emptyset$ , and  $\partial V_2 \subset J(f)$ . Note that  $W_2$  is not necessarily distinct from  $W_1$ , but that, by construction,  $V_1$  and  $V_2$  have disjoint closures.

Continuing in the same way, we can evidently construct domains  $V_1, V_2, V_3$  with pairwise disjoint closures such that, for  $j = 1, 2, 3$ ,

- $V_j \cap J(f) \neq \emptyset$ ; and
- $\partial V_j \subset J(f)$ .

Furthermore, it follows from [60, Theorem 3.3, p. 143] that we can then choose bounded, simply connected, Jordan domains  $D_1, D_2, D_3$  with pairwise disjoint closures such that  $\bar{V}_j \subset D_j$  for  $j = 1, 2, 3$ .

Everything is now in place for us to apply Proposition 2.14, a corollary of the Ahlfors five islands theorem (see Section 2.5). We therefore obtain  $\mu \in \{1, 2, 3\}$ ,  $n \in \mathbb{N}$ , and a domain  $U \subset V_\mu$  such that  $f^n : U \rightarrow D_\mu$  is conformal.

Now let  $\phi$  be the branch of the inverse function  $f^{-n}$  which maps  $D_\mu$  onto  $U$ . Then  $\phi$  must have a fixed point  $z_0 \in U \subset V_\mu$ . Furthermore, by the Schwarz lemma, this fixed point must be attracting, and because  $\phi(D_\mu) = U$  where  $\bar{U}$  is a compact subset of  $D_\mu$ , we have that  $\phi^k(z) \rightarrow z_0$  as  $k \rightarrow \infty$ , uniformly for  $z \in D_\mu$ .

Since  $z_0$  is an attracting fixed point of  $\phi$ , it is a repelling fixed point of  $f^n$  and hence a repelling periodic point of  $f$ . Thus  $z_0$  lies in  $J(f)$ .

Now  $z_0 = \phi^k(z_0) \in \phi^k(V_\mu)$  for all  $k \in \mathbb{N}$ . Furthermore,

$$\text{diam } \phi^k(\bar{V}_\mu) \rightarrow 0 \text{ as } k \rightarrow \infty.$$

It follows that

$$\bigcap_{k \in \mathbb{N}} \phi^k(\bar{V}_\mu) = \{z_0\},$$

and hence that, for any neighbourhood  $N$  of  $z_0$ , there is some  $K \in \mathbb{N}$  such that  $\phi^K(\bar{V}_\mu) \subset N$ . But  $\partial V_\mu$  lies in  $J(f)$  and  $\phi$  is conformal, so we have  $\partial \phi^K(V_\mu) = \phi^K(\partial V_\mu) \subset J(f)$ , and since  $\partial \phi^K(V_\mu)$  surrounds  $z_0$ , we have shown that an arbitrary neighbourhood  $N$  of  $z_0$  contains a continuum in  $J(f)$  that surrounds  $z_0$ .

To show that points with this property are dense in  $J(f)$ , we use the fact that  $J(f)$  is the closure of the backwards orbit  $O^-(z)$  of any point  $z \in J(f) \setminus E(f)$ . Now we may always choose our domains  $V_j$  to ensure that  $z_0 \notin E(f)$ . Therefore, since  $f$  is an open mapping and  $J(f)$  is completely invariant, it follows that each point  $z$  in the backwards

orbit  $O^-(z_0)$  has the property that any neighbourhood of  $z$  contains a continuum in  $J(f)$  that surrounds  $z$ .

Now let  $z$  be a point with this property. Evidently,  $z$  does not lie on the boundary of any component of  $F(f)$ , and so is a buried point. Let  $V$  be an open neighbourhood of  $z$  in the relative topology on  $J(f)$ , so that  $V = V' \cap J(f)$  for some open neighbourhood  $V'$  of  $z$  in  $\mathbb{C}$ . We may assume without loss of generality that  $V'$  is a disc (by making  $V$  smaller if necessary). Then it follows from the assumed property of  $z$  that  $V'$  contains a continuum  $C$  in  $J(f)$  surrounding  $z$ .

Now let  $X = \tilde{C} \cap J(f)$  (recall that  $\tilde{C}$  denotes the union of  $C$  and its bounded complementary components). Since  $J(f)$  is a spider's web, it is connected, and it follows that  $X$  is also connected. But  $\tilde{C} \subset V'$ , so  $X \subset V$ , and thus we have shown that any neighbourhood  $V$  of  $z$  in the relative topology on  $J(f)$  contains a connected neighbourhood of  $z$ . It then follows from the definition that  $J(f)$  is locally connected at  $z$ . This completes the proof.  $\square$

REMARKS. 1. If  $f$  is a transcendental entire function such that  $J(f)$  is a spider's web and  $F(f) \neq \emptyset$ , then Theorem 1.17 shows that  $J(f)$ , which is connected, contains both buried and non-buried points. This answers a question about components of the Julia set asked by Adam Epstein during a talk at the 18th International Conference on Difference Equations and Applications (Barcelona 2012). A simple example where this occurs is the sine function (see Example 5.13).

2. In Theorem 1.7, we showed that, if  $f$  is a transcendental entire function,  $R > 0$  is such that  $M(r, f) > r$  for  $r \geq R$ , and  $A_R(f)$  is a spider's web, then  $J(f)$  has a dense subset of *periodic* buried points. We remark that, using a similar method of proof, it is possible to extend Theorem 1.17 to show that, if  $f$  is a transcendental entire function such that  $J(f)$  is a spider's web, then there exists a dense subset of *periodic* buried points, at each of which  $J(f)$  is locally connected. We omit the details.

*Properties of locally  
connected  $J(f)$   
spider's webs*

Finally in this section we prove Theorem 1.18, which gives more details about  $J(f)$  for a transcendental entire function  $f$  such that  $J(f)$  is locally connected:

- (a)  $J_r(f) \neq \emptyset$ , and every neighbourhood of a point  $z \in J_r(f)$  contains a Jordan curve in  $J(f)$  surrounding  $z$ ;
- (b)  $J(f)$  is a spider's web, and there exists a sequence  $(G_n)_{n \in \mathbb{N}}$  of bounded, simply connected domains having the properties in Definition 1.1 with  $E = J(f)$ , such that the loops  $(\partial G_n)_{n \in \mathbb{N}}$  are Jordan curves.

*Proof of Theorem 1.18.* It is immediate from Theorems 1.15 and 1.17 that  $J(f)$  is a spider's web and that  $J_r(f) \neq \emptyset$ . The rest of part (a) follows from Lemma 5.2. For part (b), it remains to prove that the  $J(f)$  spider's web contains a sequence of loops that are Jordan curves.

Let  $z$  be a buried point in  $J(f)$ . Then, by part (a), there is a Jordan curve  $C$  in  $J(f)$  surrounding  $z$ . Now let  $G = \text{int}(\tilde{C})$ , and let  $\gamma_n$  be the outer boundary component of  $f^n(G)$ . Then, by the blowing up property of  $J(f)$ ,

$$\text{dist}(\gamma_n, 0) \rightarrow \infty \text{ as } n \rightarrow \infty.$$

Since  $\gamma_n \subset f^n(C)$ , it follows that  $\gamma_n$  is a Jordan curve in  $J(f)$ .

Thus  $G_n = \text{int}(\tilde{\gamma}_n)$  is a bounded Jordan domain for each  $n \in \mathbb{N}$ . Furthermore,  $\partial G_n \subset J(f)$  for each  $n \in \mathbb{N}$ , and we can choose a subsequence  $(G_{n_k})_{k \in \mathbb{N}}$  such that  $\bigcup_{k \in \mathbb{N}} G_{n_k} = \mathbb{C}$ , and  $G_{n_{k+1}} \supset G_{n_k}$  for  $k \in \mathbb{N}$ . It follows that, by relabelling  $G_{n_k}$  as  $G_k$  for  $k \in \mathbb{N}$ , we obtain a sequence of bounded Jordan domains  $(G_k)_{k \in \mathbb{N}}$  with the required properties, and this completes the proof.  $\square$

#### 5.4 THE RESIDUAL JULIA SET

In this section, we give some new results on the residual Julia set  $J_r(f)$  of a transcendental entire function  $f$ , and compare the results on  $J_r(f)$  in Theorems 1.17 and 1.18 with those obtained by other authors.

Recall that the residual Julia set  $J_r(f)$  of a mapping  $f$  is the set of buried points, i.e. the set of points in  $J(f)$  that do not lie on the boundary of any Fatou component.

First, we draw attention to a corollary of Theorem 4.2, a result due to Rippon and Stallard that we used for a different purpose in Section 4.2.

**COROLLARY 5.9.** *Let  $f$  be a transcendental entire function with non-empty residual Julia set  $J_r(f)$ . Then either  $J_r(f)$  is connected, or else  $J_r(f)$  has infinitely many components.*

*Proof.* Since  $J_r(f)$  is completely invariant and dense in  $J(f)$ , it is evident that the conditions of Theorem 4.2 hold with  $S = J_r(f)$ . Case (2) cannot occur since  $J_r(f) \cap F(f) = \emptyset$ .  $\square$

Next, we show that there are certain classes of functions for which the residual Julia set is not only connected, but is in fact a spider's web. Our result is expressed in terms of the fast escaping set  $A(f)$ .

$J_r(f)$  can be a spider's web

**THEOREM 5.10.** *Let  $f$  be a transcendental entire function, let  $R > 0$  be such that  $M(r, f) > r$  for  $r \geq R$ , and let  $A_R(f)$  be a spider's web. Assume also that  $A(f) \subset J(f)$ . Then  $J_r(f)$  is a spider's web.*

*Proof.* Since  $A(f) \cap F(f) = \emptyset$ , there are no multiply connected Fatou components by [73, Theorem 4.4], so  $J(f)$  is a spider's web by Theorem 2.8(a). Furthermore, no point on the boundary of a Fatou component of  $f$  can lie in  $A(f)$  by Theorem 2.6(b). Thus

$$A(f) \subset J_r(f) \subset J(f) = \overline{A(f)},$$

because  $J(f) = \partial A(f)$  by Theorem 2.5(c). Since  $A(f)$  is a spider's web by [73, Theorem 1.4], it follows that  $J_r(f)$  is connected and indeed is also a spider's web.  $\square$

An example of a class of functions for which  $J_r(f)$  is a spider's web is Baker's construction [7] of transcendental entire functions of arbitrarily small growth, for which every point in the Fatou set tends to a superattracting fixed point at 0 under iteration (independently, Boyd [23] arrived at a very similar construction). Clearly  $A(f) \subset J(f)$

for such functions, and it follows from Theorem 4.10(b) that  $A_R(f)$  is a spider's web.

We remark that, when  $J_\tau(f)$  is a spider's web, we have the following analogue of Theorem 1.17 (the proof is very similar and we omit it).

**THEOREM 5.11.** *Let  $f$  be a transcendental entire function such that  $J_\tau(f)$  is a spider's web. Then there exists a subset of  $J_\tau(f)$  which is dense in  $J(f)$  and consists of points  $z$  with the property that every neighbourhood of  $z$  contains a continuum in  $J_\tau(f)$  that surrounds  $z$ . At each such point,  $J_\tau(f)$  is locally connected.*

*Comparison with  
results on  $J_\tau(f)$   
obtained by other  
authors*

Finally in this section, we compare our results on  $J_\tau(f)$  in Theorems 1.17 and 1.18 with those obtained by other authors.

Theorem 1.17 gives a sufficient condition for a transcendental entire function to have  $J_\tau(f) \neq \emptyset$ , namely that  $J(f)$  is a spider's web. This complements other sufficient conditions in the literature for  $J_\tau(f)$  to be non-empty:

- Baker and Domínguez [9, Theorem 6] showed that  $J_\tau(f) \neq \emptyset$  if  $F(f)$  is not connected, there are no wandering domains, and all periodic Fatou components are bounded;
- Domínguez and Fagella [29, Proposition 6.1] proved that, if all Fatou components eventually iterate inside a closed set  $A \subsetneq \mathbb{C}$  with non-empty interior and never leave again, then  $J_\tau(f) \neq \emptyset$  provided the complement of  $A$  meets  $J(f)$ .

Theorem 1.18 gives us, in particular, that  $J_\tau(f) \neq \emptyset$  whenever  $f$  is a transcendental entire function such that  $J(f)$  is locally connected. For a general transcendental entire function, this result appears to be new. However, for transcendental entire functions in the class  $\mathcal{S}$ , it is implied by a result of Ng, Zheng and Choi [61, Theorem 2.1].

We remark that, for each of the examples given in Section 5.5 below, it is immediate from our results that the residual Julia set is not empty. This has already been proved explicitly for some of the functions or classes of functions discussed - see, for example, [29, Corollary 6.5], [58, Theorem 6] and [61, Proposition 7.1].

## 5.5 EXAMPLES

In this section we give a number of examples which illustrate the results of previous sections.

First, we consider transcendental entire functions for which the Julia set is a spider's web. We describe a large class of such functions, based on the work of Rippon and Stallard in [73]. We also show that the Julia set can be a spider's web for functions outside this class, by proving that  $J(g)$  is a spider's web when  $g(z) = \sin z$ . For each of these functions, it follows from Theorem 1.17 that the Julia set is locally connected at a dense subset of buried points.

*Transcendental  
entire functions for  
which  $J(f)$  is a  
spider's web*

In [73, Theorem 1.9], Rippon and Stallard gave many examples of functions for which the set  $A_R(f)$  is a spider's web. These examples include functions with any of the following properties:

- (a) very small growth,
- (b) order less than  $\frac{1}{2}$  and regular growth,
- (c) finite order, Fabry gaps and regular growth, or
- (d) the pits effect (as defined by Littlewood and Offord) and regular growth.

The terminology used here is defined and made precise in [73] (some of the same terminology is also discussed in Section 4.3 in connection with the proof of Theorem 1.14). Mihaljević-Brandt and Peter [55], and Sixsmith [83], have given further classes of transcendental entire functions for which  $A_R(f)$  is a spider's web.

For each of these functions, it follows from Theorem 2.8(a) that  $J(f)$  is a spider's web whenever  $f$  has no multiply connected Fatou components. Note that the escaping set  $I(f)$  is also a spider's web for these functions, by [73, Theorem 1.4].

Now a transcendental entire function such that  $A_R(f)$  is a spider's web can never belong to the class  $\mathcal{S}$  or the class  $\mathcal{B}$ , by [73, Theorem 1.8(a)]. However,  $J(f)$  can still be a spider's web in these circumstances, as we now show.

The function  $g(z) = \sin z$  has been the subject of a number of studies [8, 28, 30]. In particular, Domínguez proved in [28] that  $J(g)$  is con-



nected, and Baker and Domínguez showed in [8] that  $J(g)$  is locally connected at the fixed point 0. We now prove that  $J(g)$  is a spider's web, and also show that neither the escaping set  $I(g)$  nor the residual Julia set  $J_r(g)$  is a spider's web.

We will need the following result (see, for example, [26, Chapter 14, Theorem 7.9]).

**LEMMA 5.12** (part of the Koebe Distortion Theorem). *Let  $f$  be a function that is univalent on the unit disc with  $f(0) = 0$  and  $f'(0) = 1$ . Then, if  $|z| < 1$ ,*

$$|f(z)| \leq \frac{|z|}{(1 - |z|)^2}.$$

*The Julia set of  $\sin z$   
is a spider's web*

**EXAMPLE 5.13.** Let  $g(z) = \sin z$ . Then  $J(g)$  is a spider's web, but neither  $I(g)$  nor  $J_r(g)$  is a spider's web.

*Proof.* We first recall some basic facts about the Fatou components of  $g$  from [8, 28]. It is clear that  $g \in \mathcal{S}$ , and that the singular values of  $g$  are the two critical values at  $\pm 1$ . The fixed point 0 lies on the boundary of two invariant, parabolic Fatou components, which are reflections of one another in the imaginary axis, and in each of which  $g^n(z) \rightarrow 0$  as  $n \rightarrow \infty$ .

Label these components  $D_0$  and  $D_{-1}$ , where  $D_0$  meets the positive real axis and  $D_{-1}$  is its reflection in the imaginary axis. Then  $D_0$  and  $D_{-1}$  are bounded, and are the only two periodic Fatou components, each containing the entire orbit of one of the critical values. Furthermore, every other Fatou component is a preimage of either  $D_0$  or  $D_{-1}$  under  $g^m$  for some  $m \in \mathbb{N}$ , and the components of  $g^{-1}(D_0)$  and  $g^{-1}(D_{-1})$  all have the form

$$D_n = \{z + n\pi : z \in D_0, n \in \mathbb{Z}\}.$$

We claim that the diameters of all of the components of  $F(g)$  are uniformly bounded.

To prove the claim, we begin by using ideas from the proof of Theorem F in [8]. There it is shown that, apart from the point 0, the lemniscate  $|z^2 - 1| = 1$  lies in  $D_0 \cup D_{-1}$ . If  $h$  is any branch of  $g^{-1}$ , a straightforward calculation therefore shows that  $|h'(z)| < 1$  outside

the lemniscate, and hence in any component of  $F(g)$  other than  $D_0$  and  $D_{-1}$ .

Now let  $U$  be any Fatou component of  $g$  other than a component of  $g^{-1}(D_0)$  or  $g^{-1}(D_{-1})$ . Then there exists  $m \geq 1$  and  $n \neq 0, -1$ , such that  $g^m(U) = D_n$ . Furthermore, because the orbits of both critical values lie entirely in the real interval  $[-1, 1]$ , the branch  $\phi$  of  $g^{-m}$  mapping  $D_n$  to  $U$  is univalent in some domain  $G$  containing  $\overline{D_n}$ . Now no component of  $g^{-k}(D_n)$  for  $k \in \{1, \dots, m\}$  meets  $D_0 \cup D_{-1}$ , since  $D_0$  and  $D_{-1}$  are invariant. Therefore, since  $\phi$  is a composition of branches  $h$  of  $g^{-1}$ , for each of which  $|h'(z)| < 1$  outside  $D_0 \cup D_{-1}$ , it follows that  $|\phi'(z)| < 1$  throughout  $D_n$ .

Now, following ideas from the proof in [26, Theorem 7.16], let  $d$  be such that  $0 < 2d < \text{dist}(D_n, \partial G)$ , and cover the compact set  $\overline{D_n}$  by a finite collection  $\mathfrak{B}$  of open discs of radius  $d/8$ , each of which meets  $\overline{D_n}$ . Let  $B_1, B_2$  be two discs from this collection with non-empty intersection, and let  $z_1 \in B_1 \cap D_n$  and  $z_2 \in B_2 \cap D_n$ . Then we have  $|z_1 - z_2| < d/2$ , and  $B_1 \cup B_2 \subset \overline{B}(z_1, d) \subset G$ .

Now the function

$$\psi(z) = \frac{\phi(z_1 + dz) - \phi(z_1)}{d\phi'(z_1)}$$

is univalent in the unit disc, with  $\psi(0) = 0$  and  $\psi'(0) = 1$ . Thus it follows from Lemma 5.12 that

$$\left| \frac{\phi(z_1 + dz) - \phi(z_1)}{d\phi'(z_1)} \right| \leq \frac{|z|}{(1 - |z|)^2}$$

for  $|z| < 1$ . If we now put  $z = (z_2 - z_1)/d$ , so that  $|z| < 1/2$ , and use the fact that  $|\phi'(z)| < 1$  throughout  $D_n$ , we obtain

$$|\phi(z_2) - \phi(z_1)| \leq 2d.$$

Now let  $z, w$  be arbitrary points in  $D_n$ . Then there are points

$$z = z_1, z_2, \dots, z_k = w \text{ in } D_n, \text{ with } z_i \in B_i \in \mathfrak{B} \text{ for } i = 1, \dots, k,$$

where each consecutive pair of discs has non-empty intersection. It follows that

$$|\phi(z) - \phi(w)| \leq \sum_{j=1}^{k-1} |\phi(z_j) - \phi(z_{j+1})| \leq 2(k-1)d < 2Kd,$$

where  $K$  is the total number of discs in  $\mathfrak{B}$ . Thus the diameter of  $U$  is at most  $2Kd$ .

Furthermore, since the Fatou components  $D_n$  are congruent for all  $n \in \mathbb{Z}$ ,  $n \neq 0, -1$ , we can use the same value of  $d$  and congruent open covers whatever the value of  $n$ . Since  $D_0$  and  $D_{-1}$  are bounded, this completes the proof of the claim.

Now let  $\rho > 0$ , and let  $\{U_j : j \in \mathbb{N}\}$  be the collection of components of  $F(g)$  that meet the circle  $C(0, \rho)$ . Then it follows from the claim just proved that the set  $\bigcup_{j \in \mathbb{N}} \overline{U}_j$  is bounded. If we now let

$$X = C(0, \rho) \cup \bigcup_{j \in \mathbb{N}} \overline{U}_j,$$

and put

$$G = \text{int}(\tilde{X}),$$

we then have that  $G$  is a bounded, simply connected domain whose boundary  $\partial G$  lies in  $J(g)$ .

We can now proceed as in the proof of Theorem 5.6, and construct a sequence  $(G_k)_{k \in \mathbb{N}}$  of bounded, simply connected domains such that  $G_{k+1} \supset G_k$  and  $\partial G_k \subset J(g)$  for each  $k \in \mathbb{N}$ , and  $\bigcup_{k \in \mathbb{N}} G_k = \mathbb{C}$ . Since we know that  $J(g)$  is connected, it follows that  $J(g)$  is a spider's web.

Finally, we note that  $g$  maps the real line onto the interval  $[-1, 1]$ , so that there are no points on the real line that escape to infinity under iteration. Furthermore, all points on the real line are in the Fatou set, except for the points  $\{z = n\pi : n \in \mathbb{Z}\}$ , which each lie on the boundaries of two adjacent Fatou components. This shows that neither  $I(g)$  nor  $J_r(g)$  is a spider's web.  $\square$

Recall that, by Theorem 1.17, the Julia set for  $g(z) = \sin z$  is locally connected at a dense subset of buried points. This adds to the result of Baker and Domínguez [8] that  $J(g)$  is locally connected at the fixed

point 0 and its preimages (which are not buried points). However, it seems to be an open question whether  $J(g)$  is everywhere locally connected.

We now briefly review the conditions under which it is known that a transcendental entire function has a locally connected Julia set and give a number of examples from the literature of functions with this property. We also use results from the literature to derive some further examples. For the functions in each of these examples, it follows from Theorem 1.18 that the Julia set is a spider's web containing a sequence of loops  $(\partial G_k)_{k \in \mathbb{N}}$  which are Jordan curves and which are the boundaries of a sequence of bounded, simply connected domains  $(G_k)_{k \in \mathbb{N}}$  with the properties in Definition 1.1.

*Examples of  
transcendental  
entire functions  
with a locally  
connected Julia set*

For rational maps, it has long been known that the local connectedness of the Julia set is related to the orbits of the critical points of the map (its *critical orbits*). A rational map  $R$  is *hyperbolic* if the closure of the union of its critical orbits is disjoint from  $J(R)$  and, for such a map, if  $J(R)$  is connected then it is also locally connected. The related, but weaker, concepts of subhyperbolic, semihyperbolic and geometrically finite rational maps have also been investigated, and for these maps too, if the Julia set is connected then it is locally connected. We refer to [56, Chapter 19] and to [25, 53, 86].

Attempts to extend these ideas to transcendental entire functions have had some success. For example, the following result is a version of a theorem stated by Morosawa [58, Theorem 2].

**LEMMA 5.14.** *Let  $f$  be a transcendental entire function in the class  $\mathcal{S}$  and such that each component of  $F(f)$  contains at most finitely many critical points. Assume further that all cyclic components of  $F(f)$  are bounded. Then  $J(f)$  is locally connected if the following two conditions hold:*

- (1) *if  $\zeta \in F(f) \cap \text{sing}(f^{-1})$ , then  $\zeta$  is a critical value and is absorbed by an attracting cycle;*
- (2) *if  $\zeta \in J(f) \cap \text{sing}(f^{-1})$ , then for any Fatou component  $D$  we have*

$$\overline{\bigcup_{n \geq 0} f^n(\zeta)} \cap \partial D = \emptyset.$$

REMARK. In [58, Theorem 2] it was assumed only that  $f$  is in the class  $\mathcal{S}$ , and the additional assumption in Lemma 5.14 that each component of  $F(f)$  contains at most finitely many critical points was omitted. The proof of [58, Theorem 2] requires the deduction that if the closure of a bounded component of  $F(f)$  contains no asymptotic value of  $f$ , then all the components of its preimages are bounded. The author is grateful to the referee of the paper [63] for pointing out that this deduction requires a stronger hypothesis than that  $f$  is in the class  $\mathcal{S}$ , since a preimage could contain infinitely many critical points (in which case it must be unbounded).

Morosawa's  
examples and related  
results

Using this result, Morosawa gave the following examples of transcendental entire functions in the class  $\mathcal{S}$  for which the Julia set is locally connected (note that in each of these examples the function has only one critical point):

- $f_\lambda(z) = \lambda ze^z$ , where  $\lambda$  is such that  $f_\lambda$  has an attracting cycle of period greater than one, and satisfies  $|\operatorname{Im}(\lambda)| \geq e \operatorname{Arg}(\lambda)$  [58, Theorem 5].
- $g_\alpha(z) = \alpha e^\alpha (z - (1 - \alpha))e^z$ , where  $\alpha > 1$  [58, Theorem 7].

Indeed, Morosawa showed that  $J(g_\alpha)$  is homeomorphic to the *Sierpiński curve* continuum, i.e. that it is a nowhere dense subset of  $\widehat{\mathbb{C}}$  which is closed, connected and locally connected, and has the property that the boundaries of any two of its complementary components are disjoint Jordan curves [89]. It is a characteristic of the Sierpiński curve that it contains a homeomorphic copy of every one-dimensional plane continuum. This was explored by Garijo, Jarque and Moreno Rocha [38], who have made a detailed study of the function  $g_\alpha$ , and demonstrated the existence of indecomposable continua in its Julia set.

We note that, whenever the Julia set of a transcendental entire function is homeomorphic to the Sierpiński curve, it must necessarily also be a spider's web by Theorem 1.15.

We now use Lemma 5.14 to give the following additional example of a transcendental entire function in the class  $\mathcal{S}$  for which the Julia

set is locally connected. The example is based on work by Domínguez and Fagella [29], though they did not discuss local connectedness.

**EXAMPLE 5.15.** Let  $f(z) = \lambda \sin z$ , where  $\lambda \in \mathbb{C}$  is chosen so that there are two attracting cycles and is such that  $|\operatorname{Re}(\lambda)| \geq \frac{\pi}{2}$ . Then  $J(f)$  is locally connected.

*Proof.* It is shown in [29, Proposition 6.3] that all the Fatou components of  $f$  are bounded (note that each Fatou component contains at most one critical point). Clearly  $f \in \mathcal{S}$ , and the singular values of  $f$  are the two critical values  $\pm\lambda$ . By the choice of  $\lambda$ , each critical value is absorbed by an attracting cycle and it follows that  $J(f) \cap \operatorname{sing}(f^{-1}) = \emptyset$ . Thus conditions (1) and (2) in Lemma 5.14 hold.  $\square$

Under certain conditions, the Julia set is also locally connected for the class of semihyperbolic entire functions investigated by Bergweiler and Morosawa in [20].

*Examples due to Bergweiler and Morosawa and related results*

A transcendental entire function  $f$  is *semihyperbolic* at  $a \in J(f)$  if there exist  $r > 0$  and  $N \in \mathbb{N}$  such that, for all  $n \in \mathbb{N}$  and all components  $U$  of  $f^{-n}(B(a, r))$ , the function  $f^n|_U : U \rightarrow B(a, r)$  is a proper map of degree at most  $N$ .

Bergweiler and Morosawa's result on local connectedness is the following.

**LEMMA 5.16** (Theorem 4 in [20]). *Let  $f$  be entire. Assume that  $F(f)$  consists of finitely many attracting basins. Suppose that if  $U$  is an immediate attracting basin, then  $U$  is bounded,  $f$  is semihyperbolic on  $\partial U$ , and there exists  $N \in \mathbb{N}$  such that for every  $n \in \mathbb{N}$  and for every component  $V \neq U$  of  $f^{-n}(U) \setminus \bigcup_{k=0}^{n-1} f^{-k}(U)$  we have  $\deg(f^n|_V : V \rightarrow U) \leq N$ . Then  $J(f)$  is locally connected.*

Using this result, Bergweiler and Morosawa gave the following example of a transcendental entire function with a locally connected Julia set which is in the class  $\mathcal{B}$  but not in the class  $\mathcal{S}$ , i.e. the set  $\operatorname{sing}(f^{-1})$  is bounded but infinite.

- There exists  $A$  such that, if  $\pi^2 < a < A$ , and

$$f(z) = \frac{az}{\pi^2 - 4z} \cos \sqrt{z},$$

then  $f$  has an attracting fixed point such that  $F(f)$  consists of its basin, and the other conditions of Lemma 5.16 also hold [20, Example 2].

We have now seen examples of functions in both  $\mathcal{S}$  and in  $\mathcal{B} \setminus \mathcal{S}$  which have locally connected Julia sets. It is natural to ask for an example of a transcendental entire function  $f$  for which  $A_R(f)$  is a spider's web (so that  $f$  is in neither  $\mathcal{S}$  nor  $\mathcal{B}$ ) and  $J(f) \neq \mathbb{C}$  is locally connected. We end by using Lemma 5.16 to give such an example.

**EXAMPLE 5.17.** Let  $f$  be in the class of transcendental entire functions of arbitrarily small growth constructed by Baker in [7], for which every point in the Fatou set tends to a superattracting fixed point at 0 under iteration. Let  $R > 0$  be such that  $M(r, f) > r$  for  $r \geq R$ . Then  $A_R(f)$  is a spider's web and  $J(f)$  is locally connected.

*Proof.* It follows from Theorem 4.10 that  $A_R(f)$  is a spider's web. Furthermore, it is shown in [7] that

- (i) each component of  $F(f)$  is bounded,
- (ii)  $f^n(z) \rightarrow 0$  as  $n \rightarrow \infty$ , for each  $z \in F(f)$ ,
- (iii)  $f$  has no finite asymptotic values, and
- (iv) each of the critical points of  $f$ , other than 0, lies in the escaping set  $I(f)$ .

Let  $P(f)$  be the postcritical set of  $f$ , that is

$$P(f) = \overline{\{f^n(\zeta) : \zeta \text{ is a critical value of } f, n \geq 0\}}.$$

Then it can be shown using Baker's construction that, if  $U_0$  is the immediate basin of the superattracting fixed point 0, there is a neighbourhood  $G$  of  $U_0$  such that  $P(f) \cap G = \{0\}$ , and moreover that if  $U \neq U_0$  is any other component of  $F(f)$ , there is a neighbourhood  $G'$  of  $U$  such that  $P(f) \cap G' = \emptyset$ . We omit the details.

It follows in particular that

- $f$  is semihyperbolic on  $\partial U_0$ , and

- for each Fatou component  $U$ , there exists  $n \in \mathbb{N}$  such that  $f^n(U) = U_0$  and  $f^n|_U : U \rightarrow U_0$  is univalent.

Since  $F(f)$  consists of a single attracting basin and  $U_0$  is bounded, the conditions of Lemma 5.16 are satisfied. It follows that  $J(f)$  is locally connected.  $\square$

**REMARK.** An alternative approach to proving the local connectedness of  $J(f)$  in Example 5.17 would be to use Lemma 5.1. It follows from Theorem 1.6 that the boundary of every Fatou component of  $f$  is a Jordan curve, and a distortion argument can be used to show that, for each  $\varepsilon > 0$ , at most finitely many Fatou components have spherical diameters greater than  $\varepsilon$ .



## QUESTIONS FOR FURTHER RESEARCH

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In this brief final chapter, we identify some possible directions for future research based on the material presented in this thesis.

We begin with our results on the structure of spider's web fast escaping sets (see Theorems 1.2 to 1.7 and Chapter 3). Together with the earlier work of Rippon and Stallard described in Section 2.2, these results reveal the rich dynamical structure of the set  $A(f)^c$  for a transcendental entire function  $f$  such that  $A_R(f)$  is a spider's web. It is known that if  $A_R(f)$  is a spider's web, then so are  $A(f)$  and  $I(f)$ , and that if  $A(f)$  is a spider's web, then so is  $I(f)$  (see [73, Theorem 1.4] and the remark which follows it). However, Rippon and Stallard [75, Theorem 1.2] have recently given an example of a function  $f$  for which  $I(f)$  is a spider's web but  $A(f)$  is not, answering a question posed in [73]. The question whether  $A(f)$  can be a spider's web when  $A_R(f)$  is not a spider's web, also posed in [73], remains open. It is natural to ask the following.

*Further research on  
the structure of  
spider's web  
escaping sets*

**QUESTION 6.1.** What can we say about the dynamical structure of the set  $I(f)^c$  for a transcendental entire function  $f$  such that  $I(f)$  is a spider's web? If it transpires that  $A(f)$  can be a spider's web when  $A_R(f)$  is not, we can similarly ask what can be said about the dynamical structure of the set  $A(f)^c$  for a transcendental entire function  $f$  such that  $A(f)$  is a spider's web. In particular, are there results analogous to any of Theorems 1.2 to 1.7 in these situations?

We remark that several of our proofs in Chapter 3 rely on the strong mapping properties of the sequence of fundamental holes and loops for  $A_R(f)$  (Lemmas 3.1 and 3.3), so it is likely that any analogous results under the conditions of Question 6.1 will be weaker. However, the proof of Theorem 1.7 does not rely on these properties and we make the following conjecture.

CONJECTURE 6.2. Let  $f$  be a transcendental entire function such that  $I(f)$  is a spider's web. Then  $I(f)^c$  has singleton periodic components, and such components are dense in  $J(f)$ .

It is natural also to conjecture that the corresponding result for  $A(f)$  holds.

We turn now to the results of Section 3.4 where we used a natural partition of the plane, based on the sequence of fundamental holes and loops for  $A_R(f)$ , to encode information about orbits of points and hence of components of  $A(f)^c$ . Lemma 3.7 gives the possible mappings within the partition of the plane, and shows in particular that the mapping  $f(\overline{B}_m) = \overline{H}_{m+1}$  (3.6) holds for infinitely many values of  $m \in \mathbb{N}$ . It is not clear under what conditions the alternative mapping  $f(\overline{B}_m) = \overline{B}_{m+1}$  (3.5) holds, and it seems plausible that this will depend on the function  $f$ . We therefore ask the following.

QUESTION 6.3. Let  $f$  be a transcendental entire function, let  $R > 0$  be such that  $M(r, f) > r$  for  $r \geq R$ , and let  $A_R(f)$  be a spider's web. Under what conditions on the function  $f$  does (3.5) hold

- (a) not at all,
- (b) finitely many times, or
- (c) infinitely often,

and what additional information does this yield about the orbits of components of  $A(f)^c$ ? Using this information, is it possible to derive a function for which the orbits of particular components of  $A(f)^c$  follow a specified itinerary?

This type of itinerary is new for transcendental entire functions so there are likely to be other questions of interest concerning them.

*Further research  
on strongly  
polynomial-like  
functions*

Next, we consider our results on the connectedness properties of the set of points  $K(f)$  where the iterates of a transcendental entire function  $f$  are bounded (Theorems 1.8 to 1.14 and Chapter 4). The strongest results were obtained where  $f$  is strongly polynomial-like in the sense of Definition 1.9. In Theorem 1.10 we showed that, in this case,  $K(f)$  is either connected or has uncountably many components. At present, we know of no example of a strongly polynomial-like

function for which  $K(f)$  is either connected or has an unbounded component. Since  $K(f)$  is always unbounded, we ask the following question.

QUESTION 6.4. (a) Can  $K(f)$  be connected if  $f$  is a transcendental entire function which is strongly polynomial-like?

(b) If not, can a component of  $K(f)$  be unbounded?

It is noteworthy that strongly polynomial-like functions have some properties in common with transcendental entire functions for which  $A_R(f)$  is a spider's web (compare Theorem 1.11 with Theorems 1.5 and 1.6). It seems plausible that other properties of components of  $A(f)^c$  with bounded orbits when  $A_R(f)$  is a spider's web will also apply to components of  $K(f)$  when  $f$  is strongly polynomial-like. By analogy with Theorem 1.7 we ask:

QUESTION 6.5. Let  $f$  be a strongly polynomial-like transcendental entire function. If  $K(f)$  is disconnected, are singleton periodic components of  $K(f)$  dense in  $J(f)$ ?

Recall that if  $A_R(f)$  is a spider's web, then  $f$  is strongly polynomial-like. However, the converse is not true, as can be seen by comparing Theorem 1.14 with [75, Theorem 1.2]. Noting that  $I(f)$  is a spider's web for the function constructed in [75, Theorem 1.2], we ask the following.

QUESTION 6.6. Let  $f$  be a strongly polynomial-like transcendental entire function. Under what conditions is it true that  $I(f)$  is a spider's web? If, moreover,  $f$  has no multiply connected Fatou components, under what conditions is  $J(f)$  a spider's web?

Finally, we consider our results on the link between the spider's web form of  $J(f)$  and the property of local connectedness for transcendental entire functions (Theorems 1.15 to 1.18 and Chapter 5). In Theorem 1.17 we proved that, if  $J(f)$  is a spider's web, then there is a dense subset of buried points at which  $J(f)$  is locally connected. It is natural to ask whether or not  $J(f)$  is then necessarily locally connected at all of its points.

*Further research on  
local connectedness*

QUESTION 6.7. Let  $f$  be a transcendental entire function such that  $J(f)$  is a spider's web. Can  $J(f)$  fail to be locally connected at any of its points?

A related question is whether  $J(g)$  is everywhere locally connected for the function  $g(z) = \sin z$  (see Example 5.13). If it is not, this will obviously settle Question 6.7.

The results in Theorems 1.15 to 1.18 concern the local connectedness of the Julia set of a transcendental entire function, but they suggest that there may be a link between the spider's web form of other dynamically important sets and the property of local connectedness. We therefore ask the following.

QUESTION 6.8. Is there a link between the spider's web form of the set  $S$  and the property of local connectedness, where  $S$  is

$$A_R(f) \cap J(f), A_R(f), A(f) \cap J(f), A(f), I(f) \cap J(f) \text{ or } I(f).$$

In particular:

- (a) If  $S$  is locally connected, is  $S$  a spider's web?
- (b) If  $S$  is a spider's web, is there a dense subset of points at which  $S$  is locally connected?

We remark that our proofs in Theorems 1.15 to 1.18 rely on the 'blowing up' property of the Julia set (Lemma 2.2(d)), and it is therefore perhaps more likely that the answer to Question 6.8 will be positive for the sets  $A_R(f) \cap J(f)$ ,  $A(f) \cap J(f)$  and  $I(f) \cap J(f)$  than for the sets  $A_R(f)$ ,  $A(f)$  and  $I(f)$ .

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